

620-142 Mathematics B
Solutions to assignment 3

1. Reducing the matrix A to row echelon form gives

$$\begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 2 & 2 & 4 & -2 & 2 \\ 3 & 3 & 6 & 5 & 11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus:

- (a) $\text{rank } A = 2$,
- (b) The column space has a basis made up of the columns of A corresponding to the leading entries of the row echelon form. So a basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \right\}$ and $\dim = 2$.
- (c) The row space has a basis made up of the non-zero rows in any row echelon form. So one possible basis is $\{(1, 1, 2, -1, 1), (0, 0, 0, 1, 1)\}$ and $\dim = 2$.
- (d) The general solution to $A\mathbf{x} = \mathbf{0}$ has 3 parameters $x_2 = s, x_3 = t, x_5 = u$ and $x_4 = -x_5 = -u, x_1 = -x_2 - 2x_3 - 2x_5 = -s - 2t - 2u$. So the general solution is

$$(x_1, x_2, x_3, x_4, x_5) = (-s - 2t - 2u, s, t, -u, u) \text{ where } s, t, u \in \mathbb{R}.$$

Thus the solution space has a basis

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

and its dimension is 3.

(**Note:** many other bases are also possible in parts (b)-(d).)

2. (a) This is **not** a subspace as it does not contain the zero vector: the 2×2 zero matrix has $\det = 0$ so is not in H .
- (b) This **is** a subspace as it satisfies the three subspace conditions:
- (0) H is non-empty as it contains the zero polynomial.
- (1) H is closed under addition: If p and q are polynomials in H , then $p(3) = 0$ and $q(3) = 0$. Hence $(p + q)(3) = p(3) + q(3) = 0 + 0 = 0$, and $p + q \in H$.
- (2) H is closed under scalar multiplication: If $p \in H$ and $c \in \mathbb{R}$, then $(cp)(3) = cp(3) = c \cdot 0 = 0$, so $cp \in H$.
3. (a) We need to solve $\alpha A + \beta B + \gamma C = \mathbf{0}$, where α, β, γ are scalars and $\mathbf{0}$ is the 2×2 zero matrix. This is the matrix equation:

$$\begin{bmatrix} \alpha - \beta - \gamma & 2\alpha + 4\beta + \gamma \\ 3\alpha + 2\beta + \gamma & -\alpha + \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equating matrix entries gives a system of 4 homogeneous linear equations. Reducing the coefficient matrix to row echelon form gives

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 6 & 3 \\ 0 & 5 & 4 \\ 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This has rank 3 = number of unknowns, so the only solution is $\alpha = \beta = \gamma = 0$ and the vectors are **linearly independent**.

(b) We try to solve

$$\alpha(1 - x + x^2) + \beta(3 - x^2) + \gamma(1 + 2x - 3x^2) = a + bx + cx^2$$

or

$$(\alpha + 3\beta + \gamma) + (-\alpha + 2\gamma)x + (\alpha - \beta - 3\gamma)x^2 = a + bx + cx^2$$

where a, b, c are arbitrary real numbers. Equating coefficients of $1, x, x^2$ gives a system of 3 linear equations. Reducing the coefficient matrix to row echelon form gives

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & 0 & 2 & b \\ 1 & -1 & -3 & c \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 3 & 3 & a+b \\ 0 & -4 & -4 & c-a \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 1 & \frac{1}{3}(a+b) \\ 0 & 0 & 0 & \frac{1}{3}a + \frac{4}{3}b + c \end{array} \right].$$

So we **cannot** solve the system for *all* choices of a, b, c , and the polynomials **do not** span \mathcal{P}_2 .

(c) The number of vectors in any linearly independent set is at most $\dim \mathcal{P}_n = n + 1$. Hence $k \leq n + 1$.

4. (a) (i) The check digit x_{11} satisfies $0 - 3 + 6 - 9 + 1 - 4 + 7 - 2 + 5 - 8 + x_{11} \equiv 0 \pmod{10}$. Hence $-7 + x_{11} \equiv 0 \pmod{10}$, and $x_{11} = 7$.
- (ii) Interchanging two adjacent digits differing by 5 gives the same sum modulo 10. (For example, changing x_1x_2 to x_2x_1 changes the sum by $(x_2 - x_1) - (x_1 - x_2) = 2(x_2 - x_1) \equiv 0 \pmod{10}$ if $x_2 - x_1 = \pm 5$.)

(b) (i) The coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is already in row echelon form so we have parameters $x_4 = s, x_5 = t, x_6 = u$ and $x_1 = -x_4 - x_5 = x_4 + x_5 = s + t, x_2 = x_5 + x_6 = t + u, x_3 = x_4 + x_5 + x_6 = s + t + u$. Thus the general solution is

$$\mathbf{x} = (s+t, t+u, s+t+u, s, t, u) = s(1, 0, 1, 1, 0, 0) + t(1, 1, 1, 0, 1, 0) + u(0, 1, 1, 0, 0, 1),$$

where $s, t, u \in \mathbb{Z}_2 = \{0, 1\}$, and a basis for the solution space is

$$\{(1, 0, 1, 1, 0, 0), (1, 1, 1, 0, 1, 0), (0, 1, 1, 0, 0, 1)\}.$$

(ii) Choosing $stu = 000, 001, 010, 011, 100, 101, 110, 111$ gives the $2^3 = 8$ solutions:

$$000000, 011001, 111010, 100011, 101100, 110101, 010110, 001111,$$

where we write $x_1x_2x_3x_4x_5x_6$ as an abbreviation for $(x_1, x_2, x_3, x_4, x_5, x_6)$.

(iii) To detect errors, we first multiply the check matrix by the received messages (written as column vectors):

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For the first message the result is the 4th column of the check matrix, so there is an error in the 4th bit, and the corrected message is 111010.

For the second message the result is non-zero and not a column of the check matrix, so more than one error has occurred.