

620-142 Mathematics B
Assignment 4
Due: 2pm, Friday, May 16

Please leave your assignment in your tutor's box located near the north entrance to the Richard Berry building. Make sure that you have written your name, your student number, your tutor's name, and your tutorial time on the front page.

You should give **complete explanations** for all your answers.

1. (a) If $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ are polynomials in \mathcal{P}_2 we define

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Show that this is an inner product on \mathcal{P}_2 .

- (b) We define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_3v_3$$

for vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in \mathbb{R}^3 .

Explain the reason(s) why this is **not** an inner product on \mathbb{R}^3 .

- (c) Let V denote the vector space of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. We define an inner product on V by putting

$$\langle f, g \rangle = \int_0^1 xf(x)g(x) dx.$$

If $f(x) = x$ and $g(x) = x^2 + 1$, use this inner product to find:

- (i) $\langle f, g \rangle$ and $\langle f, f \rangle$,
 - (ii) $\|f\|$,
 - (iii) all values of $\alpha \in \mathbb{R}$ such that $g + \alpha f$ is orthogonal to f in V .
2. (a) Let W be the subspace of \mathbb{R}^4 with basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1, 0, 0, 1), \quad \mathbf{v}_2 = (3, 2, 0, 1) \quad \text{and} \quad \mathbf{v}_3 = (1, 1, 2, -3).$$

Use the Gram-Schmidt process to transform S to an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for W , using the dot product as inner product. (Marks will be allocated for checking that \mathbf{u}_1 , \mathbf{u}_2 and \mathbf{u}_3 are in fact orthogonal.)

- (b) You may assume that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for \mathbb{R}^3 using the Euclidean inner product, where

$$\mathbf{u}_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \quad \mathbf{u}_2 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \quad \text{and} \quad \mathbf{u}_3 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right).$$

Write the vector $\mathbf{v} = (0, 1, 5)$ as a linear combination of the vectors \mathbf{u}_1 , \mathbf{u}_2 and \mathbf{u}_3 .

Please turn over.

3. The following table shows the winning times in the women's 100m freestyle final at the Olympic Games from 1956 to 2000. Here x is the number of years since 1900 and y is the time in seconds.

x	56	60	64	68	72	76	80	84	88	92	96	100
y	62.0	61.2	59.5	60.0	58.59	55.65	54.79	55.92	54.93	54.65	54.50	53.83

Use the method of least squares to answer the following.

- Find the straight line $y = a + bx$ which best fits this data, and use your answer to predict the winning time at the 2004 Olympics.
- Find the quadratic equation $y = a + bx + cx^2$ which best fits this data, and use your answer to predict the winning time at the 2004 Olympics.
(The actual winning time in 2004 was 53.84 seconds by Jodie Henry.)

(If you wish, you may use MATLAB to solve the normal systems that arise in this question, but you should explain how these normal systems are obtained and indicate the MATLAB commands you used.)

Challenge problem (Not for assessment)

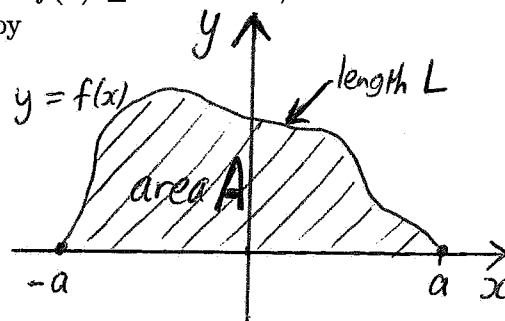
(Chocolate bars will be given for the best solutions!)

Let $a > 0$ and let $f : [-a, a] \rightarrow \mathbb{R}$ be a function¹ such that $f(x) \geq 0$ for all x , and $f(-a) = f(a) = 0$. Then the area under the graph of f is given by

$$A = \int_{-a}^a f(x) dx$$

and the length of the graph of f is

$$L = \int_{-a}^a \sqrt{1 + f'(x)^2} dx.$$



- (a) Show that

$$A = \int_{-a}^a -x f'(x) dx \quad \text{and} \quad \frac{\pi a^2}{2} = \int_{-a}^a \sqrt{a^2 - x^2} dx.$$

- (b) Prove that

$$A + \frac{\pi a^2}{2} \leq aL.$$

(Hint: the Cauchy-Schwarz inequality may be useful.)

- (c) Deduce that

$$A \leq \frac{L^2}{2\pi}.$$

- (d) Show that

$$A = \frac{L^2}{2\pi} \quad \text{if and only if} \quad f(x) = \sqrt{a^2 - x^2},$$

i.e. if and only if the graph is a semi-circle.

- (e) How is this question related to Queen Dido and the founding of Carthage?

¹You may assume f is continuous on $[-a, a]$ and differentiable on $(-a, a)$.