620-142 Mathematics B
Solutions to Assignment 4

1. (a) We check the innner product axioms. For any polynomials $\mathbf{p}, \mathbf{q}, \mathbf{r}$ in $\mathcal{P}_2$ and any scalar $c \in \mathbb{R}$ we have:

   1. $\langle \mathbf{q}, \mathbf{p} \rangle = q(0)p(0) + q(1)p(1) + q(2)p(2) = p(0)q(0) + p(1)q(1) + p(2)q(2) = \langle \mathbf{p}, \mathbf{q} \rangle$

   2. $\langle c\mathbf{p}, \mathbf{q} \rangle = cp(0)q(0) + cp(1)q(1) + cp(2)q(2) = c\langle \mathbf{p}, \mathbf{q} \rangle$

   $$\begin{aligned} 3.\ \langle \mathbf{p}+\mathbf{q}, \mathbf{r} \rangle &= (p(0)+q(0))r(0) + (p(1)+q(1))r(1) + (p(2)+q(2))r(2) \\ &= (p(0)r(0) + p(1)r(1) + p(2)r(2)) + (q(0)r(0) + q(1)r(1) + q(2)r(2)) \\ &= \langle \mathbf{p}, \mathbf{r} \rangle + \langle \mathbf{q}, \mathbf{r} \rangle \end{aligned}$$

   For 4(a), note that
   $$\langle \mathbf{p}, \mathbf{p} \rangle = p(0)^2 + p(1)^2 + p(2)^2 \geq 0.$$

   For 4(b), note that $\langle \mathbf{p}, \mathbf{p} \rangle = 0$ implies $p(0) = p(1) = p(2) = 0$. If $p(x) = a + bx + cx^2$ this means that $a = 0$, $a + b + c = 0$ and $a + 2b + 4c = 0$. Solving these linear equations gives $a = b = c = 0$, hence $\mathbf{p}$ is the zero polynomial.
   (Or: use the fact that a polynomial of degree ≤ 2 vanishing at three points must be the zero polynomial.)

   (b) This fails the positivity condition 4(b): For example, if $\mathbf{u} = (0, 1, 0)$, then $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ but $\mathbf{u} \neq \mathbf{0}$.

   (c) We have $\langle f, g \rangle = \int_0^1 x f(x) g(x)\, dx$ for all continuous functions f, g.

   (i) Taking $f(x) = x$, $g(x) = x^2 + 1$ in the above definition gives:

   $$\langle f, g \rangle = \int_0^1 x \cdot x \cdot (x^2+1)\, dx = \int_0^1 (x^4 + x^2)\, dx = \left[\frac{x^5}{5} + \frac{x^3}{3}\right]_0^1 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15},$$

   $$\langle f, f \rangle = \int_0^1 x \cdot x \cdot x\, dx = \int_0^1 x^3\, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4},$$

   (ii) $\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

   (iii) From the inner product axioms and part (i) (or direct calculation):

   $$\langle f, g + \alpha f \rangle = \langle f, g \rangle + \alpha \langle f, f \rangle = \frac{8}{15} + \frac{1}{4}\alpha.$$

   The functions $g + \alpha f$ and f are orthogonal if this inner product is zero. Thus

   $$\frac{8}{15} + \frac{1}{4}\alpha = 0, \quad \text{and} \quad \alpha = -\frac{32}{15}.$$

2. (a) To find an orthonormal basis for W we apply the Gram-Schmidt procedure to the vectors $\mathbf{v}_1 = (1,0,0,1)$, $\mathbf{v}_2 = (3,2,0,1)$ and $\mathbf{v}_3 = (1,1,2,-3)$ (with inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w}$).

$$\begin{aligned}
\mathbf{v}_1 &= (1,0,0,1), \qquad ||\mathbf{v}_1||^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1+0+0+1 = 2, \text{ so} \\
\mathbf{u}_1 &= \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = \frac{1}{\sqrt{2}}(1,0,0,1).
\end{aligned}$$

$$\begin{aligned}
\mathbf{w}_2 &= \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 \\
&= (3,2,0,1) - \frac{3+0+0+1}{2}(1,0,0,1) \\
&= (3,2,0,1) - (2,0,0,2) = (1,2,0,-1) \\
\mathbf{u}_2 &= \frac{\mathbf{w}_2}{||\mathbf{w}_2||} = \frac{1}{\sqrt{6}}(1,2,0,-1).
\end{aligned}$$

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2 \\
&= (1,1,2,-3) - \frac{1+0+0-3}{2}(1,0,0,1) - \frac{(1+2+0+3)}{6}(1,2,0,-1) \\
&= (1,1,2,-3) + (1,0,0,1) - (1,2,0,-1) \\
&= (1,-1,2,-1), \\
\mathbf{u}_3 &= \frac{\mathbf{w}_3}{||\mathbf{w}_3||} = \frac{1}{\sqrt{7}}(1,-1,2,-1).
\end{aligned}$$

Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for W. As a (partial) check we note that

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{1}{\sqrt{2}}(1,0,0,1) \cdot \frac{1}{\sqrt{6}}(1,2,0,-1) = 0, \\
\mathbf{u}_1 \cdot \mathbf{u}_3 &= \frac{1}{\sqrt{2}}(1,0,0,1) \cdot \frac{1}{\sqrt{7}}(1,-1,2,-1) = 0, \\
\mathbf{u}_2 \cdot \mathbf{u}_3 &= \frac{1}{\sqrt{6}}(1,2,0,-1) \cdot \frac{1}{\sqrt{7}}(1,-1,2,-1) = 0.
\end{aligned}$$

So the vectors are orthogonal as desired.

(b) Since the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ form an orthonormal basis for $\mathbb{R}^3$ we have

$$\begin{aligned}
\mathbf{v} &= \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \langle \mathbf{v}, \mathbf{u}_3 \rangle \mathbf{u}_3 \\
&= \left[(0,1,5) \cdot (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\right] \mathbf{u}_1 + \left[(0,1,5) \cdot (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})\right] \mathbf{u}_2 + \left[(0,1,5) \cdot (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})\right] \mathbf{u}_3 \\
&= 4\mathbf{u}_1 + 1\mathbf{u}_2 - 3\mathbf{u}_3.
\end{aligned}$$

3. (a) We want to fit a straight line $y = a + bx$ to the data. Using the method of least squares the unknown coefficients a, b are determined by the normal equation

$$A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T \vec{y},$$

where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{12} \end{bmatrix} = \begin{bmatrix} 1 & 56 \\ 1 & 60 \\ \vdots & \vdots \\ 1 & 100 \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{12} \end{bmatrix} = \begin{bmatrix} 62.0 \\ 61.2 \\ \vdots \\ 53.83 \end{bmatrix}.$$

We use MATLAB to solve this equation. Using the commands:
$x = [\ 56\ \ 60\ \ 64\ \ 68\ \ 72\ \ 76\ \ 80\ \ 84\ \ 88\ \ 92\ \ 96\ \ 100\ ]'$

$y = [\ 62.0\ 61.2\ 59.5\ 60.0\ 58.59\ 55.65\ 54.79\ 55.92\ 54.93\ 54.65\ 54.50\ 53.83\ ]'$
$A = [x. \wedge 0\ \ x. \wedge 1]$
$\mathrm{rref}([A' * A\ \ A' * y])$
gives output

```
1.0000      0      72.0168
     0 1.0000     −0.1909
```

and we read off the coefficients $a = 72.0168$, $b = -0.1909$ from the last column. Hence the least squares line of best fit has equation

$$y = 72.0168 - 0.1909x.$$

To predict the 2004 winning time we take $x = 104$ in the line of best fit, giving $y \approx 52.16$ seconds.

(b) To find the quadratic equation $y = a + bx + cx^2$ which best fits the data, we solve

$$A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \vec{y}, \text{ where } A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_{12} & x_{12}^2 \end{bmatrix}.$$

Using the MATLAB commands:
$A = [x. \wedge 0\ \ x. \wedge 1\ \ x. \wedge 2]$
$\mathrm{rref}([A' * A\ \ A' * y])$
gives output

```
1.0000      0      0     93.8464
     0 1.0000      0     −0.7687
     0      0 1.0000      0.0037
```

and we read off the coefficients a, b, c from the last column. Hence the quadratic equation of best fit is

$$y = 93.8464 - 0.7687x + 0.0037x^2.$$

To predict the 2004 winning time we take $x = 104$ in this quadratic equation, giving $y \approx 53.92$ seconds. (Quite an accurate estimate!)