**620-142 Mathematics B**
**Assignment 5**
**Due: 2pm, Friday, May 30**

Please leave your assignment in your tutor's box located near the north entrance to the Richard Berry building. Make sure that you have written your name, your student number, your tutor's name, and your tutorial time on the front page.

You should give **complete explanations** for all your answers.

1. Determine whether the following are linear transformations. Explain your answers.

   (a) $T : M^{2,2} \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

   (b) $S : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $S(a_0 + a_1x + a_2x^2) = a_0x + (a_1 + a_2)x^3$

2. Find a single matrix that performs the following sequence of operations in the plane: reflection in the line $y = -x$, then rotation through an angle of $\dfrac{\pi}{3}$ radians anticlockwise around the origin.

3. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation

$$T(x, y, z) = (x - z, 2x + y, x + 2y + 3z).$$

   (a) Write down the standard matrix for $T$.

   (b) Find a basis for the kernel of $T$.

   (c) Find a basis for the image of $T$.

   (d) Write down the rank and the nullity of $T$.

4. Let $\mathcal{P}_3$ denote the vector space of all real polynomials of degree $\leq 3$ in the variable $x$. A linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ is defined by

$$T(f(x)) = f'(x) + 3f(x).$$

   Find the matrix which represents $T$ relative to the basis $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3$.

5. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}.$$

   (a) Write down the transition matrix from $B'$ to $B$.

   (b) Find the transition matrix from $B$ to $B'$.

   (c) Find $[\mathbf{u}_3]_{B'}$.

6. (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

   (b) Find a diagonal matrix $D$ and invertible matrices $P, P^{-1}$ such that $P^{-1}AP = D$.

   (c) Use your answers from (b) to find $A^n$ for each integer $n \geq 1$.

*Please turn over for Challenge Problem.*

**Challenge problem (Not for assessment)**
(Edible prizes will be given for the best solutions!)

Let $A$ be a $2 \times 2$ matrix with **integer** entries such that

$$A^k = I = \text{ identity}$$

for some positive integer $k$, and assume that no smaller $k$ has this property. (Then $k$ is called the *order* of $A$.)

(a) Show that $\det(A) = \pm 1$.

(b) Show that each eigenvalue $\lambda$ of $A$ satisfies $\lambda^k = 1$.

(c) Show that the product of the eigenvalues of $A$ is $\pm 1$ and the sum of the eigenvalues of $A$ is an integer.

(d) Determine all the possible values of $k$, and give a corresponding matrix $A$ for each $k$. [Hint: use parts (a)–(c) to determine the possible eigenvalues of $A$.]

(e) Try to generalize your results to the $n \times n$ case.

[Remark: the values of $k$ obtained in part (d) are the possible orders of symmetry of 2-dimensional crystalline lattices.]