620-142 Mathematics B
Solutions to Assignment 5

1. (a) T is not a linear transformation. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $T(A) = \det(A) = 1$ but $T(2A) = 4$. So $T(2A) \neq 2T(A)$.

   (b) S is a linear transformation. Let $p = a_0 + a_1x + a_2x^2, q = b_0 + b_1x + b_2x^2 \in \mathcal{P}_2$ and $c \in \mathbb{R}$. Then $p + q = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$, so

$$S(p+q) = (a_0+b_0)x+(a_1+b_1+a_2+b_2)x^3 = a_0x+(a_1+a_2)x^3+b_0x+(b_1+b_2)x^3 = S(p)+S(q)$$

   and $cp = ca_0 + ca_1x + ca_2x^2$ so

$$S(cp) = ca_0x + (ca_1 + ca_2)x^3 = c(a_0x + (a_1 + a_2)x^3) = cS(p).$$

2. Reflection in $y = -x$ takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and takes $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so has standard matrix

$$S = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

   Rotation by $\frac{\pi}{3}$ anticlockwise has standard matrix

$$T = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

   Hence the combined transformation has matrix

$$TS = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

3. (a) The standard matrix for T is

$$A_T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

   (b) The kernel of T is the nullspace of A_T.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   so the general solution is $x_3 = t, x_2 = -2t, x_1 = t$ Hence, a basis for the kernel of T is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

   (c) A basis for the image of T = column space of A_T is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

   (d) rank = dim(image)= 2, nullity = dim(kernel) =1

4. We have $T(1) = 0 + 3 = 3$, $T(x) = 1 + 3x$, $T(x^2) = 2x + 3x^2$, $T(x^3) = 3x^2 + 3x^3$. Hence the matrix of T relative to the basis $B = \{1, x, x^2, x^3\}$ is

$$[T]_B = \big[[T(1)]_B \, [T(x)]_B \, [T(x^2)]_B \, [T(x^3)\,]_B \big] = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

5. (a) Since B is the standard basis we have

$$P_{B'\to B} = [\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

(b) We calculate $P_{B\to B'} = (P_{B'\to B})^{-1}$ using the Gauss-Jordan method. By row reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & -3 \\ 0 & 1 & 0 & 6 & -3 & 4 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

So

$$P_{B\to B'} = \begin{bmatrix} -5 & 3 & -3 \\ 6 & -3 & 4 \\ 2 & -1 & 1 \end{bmatrix}.$$

(c)

$$[\mathbf{u}_3]_{B'} = P_{B\to B'}[\mathbf{u}_3]_B = \begin{bmatrix} -5 & 3 & -3 \\ 6 & -3 & 4 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

6. (a) The eigenvalues satisfy

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6).$$

So the eigenvalues are $\lambda = 1, 6$.
Eigenvectors for $\lambda = 1$ satisfy $(A - I)\mathbf{x} = \mathbf{0}$. Now

$$A + 3I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

So $x_2 = -2x_1$ and an eigenvector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
Eigenvectors for $\lambda = 6$ satisfy $(A - 6I)\mathbf{x} = \mathbf{0}$. Now

$$A - 2I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

So $x_1 = 2x_2$ and an eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b)

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

(c) $A = PDP^{-1}$ hence

$$\begin{aligned} A^n &= PD^nP^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 6^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 1 & 2\cdot 6^n \\ -2 & 6^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 1 + 4\cdot 6^n & -2 + 2\cdot 6^n \\ -2 + 2\cdot 6^n & 4 + 6^n \end{bmatrix}. \end{aligned}$$

Challenge problem

Let A be a 2×2 matrix with **integer** entries such that

$$A^k = I = \text{ identity}$$

for some positive integer k, and assume that no smaller k has this property. (Then k is called the *order* of A.)

(a) $1 = \det(A^k) = \det(A)^k$ and $\det(A)$ is a integer since A has integer entries, so $\det(A) = \pm 1$.

(b) Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with eigenvector $\mathbf{v} \in \mathbb{C}^n$. Then $A\mathbf{v} = \lambda\mathbf{v}$ and, by induction, $A^n\mathbf{v} = \lambda^n\mathbf{v}$ for all integers $n \geq 1$. Since $A^k = I$ we have $\mathbf{v} = A^k\mathbf{v} = \lambda^k\mathbf{v}$ and since $\mathbf{v} \neq 0$ it follows that $\lambda^k = 1$.

(c) Form lectures we know that the product of the eigenvalues of A is $\det(A) = \pm 1$ and the sum of the eigenvalues of A is trace (A) so is an integer.

(d) The eigenvalues λ_1, λ_2 satisfy a quadratic equation $\det(A - \lambda I) = 0$ with real coefficients, so either both eigenvalues are real or the eigenvalues are complex conjugates.

If the eigenvalues are real, then $\lambda_1, \lambda_2 = \pm 1$ and it follows that A has order 1 or 2.

If the eigenvalues are complex conjugates, $\lambda_1 = \exp(2\pi i n/k), \lambda_2 = \exp(-2\pi i n/k)$ for some integer n and $\lambda_1 + \lambda_2 = 2\cos(2\pi n/k)$ is an integer. Hence $2\cos(2\pi n/k) = -2, -1, 0, 1$ or 2, and it follows that $k = 2, 3, 4, 6$ or 1.

So the possible orders of A are $1, 2, 3, 4, 6$. These occur, for example, for the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

(These correspond to rotational symmetries of a square or regular hexagonal lattice in the plane.)

(e) I'll leave this for you to try...