

The University of Melbourne
Semester Two 2004
Department of Mathematics and Statistics
620-142 Mathematics B

Exam duration: Three hours

Reading time: 15 minutes

This paper has 7 pages

Common Content: This examination paper contains questions in common with the paper for 620-122.

Authorized Materials: No materials are authorized. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators: One 14 page script book is to be given to each student initially. Students may retain this examination paper. No written or printed material related to the subject may be brought into the examination. No mathematical tables or calculators may be used.

Instructions to Students: This examination consists of 12 questions. All questions may be attempted. The number of marks for each question is indicated on the examination paper. The total number of marks is 90. Use of calculators is not allowed.

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1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & -2 \\ -5 & -10 & -19 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

You are given that the matrix B is the reduced row echelon form of the matrix A . Using this information, or otherwise, answer the following.

- (a) What is the rank of A ?
- (b) Write down a basis for the column space of A .
- (c) Find the dimension of the row space of A .
- (d) Are the rows of A linearly independent? Explain your answer.
- (e) Do the vectors $(1, 2, -5)$, $(3, 8, -19)$, $(1, -2, 3)$ span \mathbb{R}^3 ? Give a reason.
- (f) Write $(1, -2, 3)$ as a linear combination of $(1, 2, -5)$ and $(3, 8, -19)$.
- (g) Use your answer to (a) to compute the dimension of the solution space of A .
- (h) Find a basis for the solution space of A .

[11 marks]

2. Determine which of the following are subspaces of the vector space \mathcal{P}_2 of polynomials in x of degree at most 2. Explain your answers by either verifying the appropriate axioms, or providing a counter example.

- (a) The set of polynomials whose coefficient of x^2 is non-negative:

$$A = \{a + bx + cx^2 : c \geq 0\}$$

- (b) The set of polynomials vanishing at $x = 1$:

$$B = \{p(x) : p(1) = 0\}$$

[5 marks]

Please turn over ...

3. Consider the linear code with check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

where the entries are in \mathbb{Z}_2 .

- (a) The messages 1100110 and 0100101 are received.
 - (i) Which of the messages must contain an error?
 - (ii) Find the original messages, assuming at most one error occurred.
- (b) Write down the dimension of the solution space of H using modulo 2 arithmetic. Hence write down the number of codewords.

[5 marks]

4. For column vectors \mathbf{x}, \mathbf{y} in \mathbb{R}^2 we define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$ where

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

From lectures we know that $\langle \cdot, \cdot \rangle$ automatically satisfies the two linearity properties for an inner product.

- (a) Show that $\langle \cdot, \cdot \rangle$ is
 - (i) symmetric (i.e. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all \mathbf{x}, \mathbf{y}), and
 - (ii) positive definite (i.e. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for all \mathbf{x} and $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = \mathbf{0}$).
Therefore $\langle \cdot, \cdot \rangle$ defines an inner product on \mathbb{R}^2 .
- (b) Show that

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is an orthonormal basis for \mathbb{R}^2 with respect to this inner product.

- (c) Compute the length $\|\mathbf{y}\|$ of $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with respect to the above inner product.
- (d) Let $W = \text{span}\{\mathbf{u}_2\}$ be the subspace spanned by the vector \mathbf{u}_2 . Compute $\mathbf{p} = \text{proj}_W \mathbf{y}$, the orthogonal projection of the vector \mathbf{y} onto W , with respect to the above inner product.

[9 marks]

Please turn over ...

5. (a) Use the method of least squares to find an equation $y = a + bx$ which best fits the following data.

x	-1	0	1	2
y	4	2	0	-4

- (b) Check your answer by plotting the data points and your straight line.
- (c) Use your line of best fit to estimate the value of y when $x = -2$.
[6 marks]
6. (a) Let $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection in the y -axis and let $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation by 90 degrees anticlockwise about the origin.
- (i) Write down the standard matrix representations for S and R .
- (ii) Use part (i) to find the standard matrix representation for $S \circ R$ (R followed by S).
- (iii) Draw the triangle T with corners at $(0, 0)$, $(1, 0)$, $(1, 1)$ in the xy -plane and on the same set of axes draw the image of this triangle T' after applying $S \circ R$.
- (iv) Give a geometric interpretation of $S \circ R$.
[5 marks]

- (b) Let $M^{2,2}$ be the vector space of real 2×2 matrices. Let $S : M^{2,2} \rightarrow M^{2,2}$ be the linear transformation given by

$$S(A) = A - A^T$$

where A^T is the transpose of the 2×2 matrix $A \in M^{2,2}$.

- (i) Compute the 4×4 matrix $[S]_{\mathcal{B}}$ for S with respect to the standard basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

- (ii) Using part (i) or otherwise, find the dimension of the image of S .
[4 marks]

Please turn over ...

7. Let A be the 2×2 real matrix

$$A = \begin{bmatrix} \frac{5}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

- (a) Compute the eigenvalues and corresponding eigenvectors of A .
- (b) Find invertible matrices P , P^{-1} and a diagonal matrix D such that $A = PDP^{-1}$.
- (c) Give a formula for A^n in terms of P , D and P^{-1} .
- (d) Compute $\lim_{n \rightarrow \infty} A^n$.

[8 marks]

8. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

- (a) Calculate the eigenvalues of A .
- (b) Find corresponding unit eigenvectors of A .
- (c) Find an orthogonal matrix Q and a diagonal matrix D such that

$$D = Q^T A Q.$$

[8 marks]

Please turn over ...

In the following questions you may use the following standard limits. Note that $\log n$ denotes the natural logarithm of n .

$$\begin{array}{ll} \text{(i)} & \lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \quad (p > 0) \\ \text{(ii)} & \lim_{n \rightarrow \infty} r^n = 0 \quad (|r| < 1) \\ \text{(iii)} & \lim_{n \rightarrow \infty} a^{1/n} = 1 \quad (a > 0) \\ \text{(iv)} & \lim_{n \rightarrow \infty} n^{1/n} = 1 \\ \text{(v)} & \lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0 \quad (\text{all } a) \\ \text{(vi)} & \lim_{n \rightarrow \infty} \frac{\log n}{n^p} = 0 \quad (p > 0) \\ \text{(vii)} & \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = e^a \\ \text{(viii)} & \lim_{n \rightarrow \infty} \frac{n^p}{a^n} = 0 \quad (p > 0, a > 1) \end{array}$$

You may also find the following ordering by growth rates useful:

$$c \ll \log n \ll n^p \ll a^n \ll b^n \ll n! \ll n^n$$

where c is any constant, $p > 0$ and $1 < a < b$.

9. Determine which of the following sequences converge and which diverge. Find the limit for each convergent sequence. Please give complete arguments using above the growth rates, standard limits, arithmetic of limits, and other relevant theorems.

$$\begin{array}{l} \text{(a)} \quad \frac{\log n + e^n}{2^n + n^2} \\ \text{(b)} \quad \frac{3n + \sin(n)}{n + 7} \\ \text{(c)} \quad \frac{\log(n)}{\log(n + 1)} \\ \text{(d)} \quad a_{n+1} = \frac{2}{3} \left(a_n + \frac{1}{a_n^2} \right) \text{ and } a_1 = 2. \text{ [You may assume that this limit exists.]} \end{array}$$

[9 marks]

Please turn over ...

10. Determine which of the following series converge and which diverge. Please justify your conclusions by referring to appropriate tests.

$$(a) \sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$(c) \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \dots$$

[9 marks]

11. The function $\log(1+x)$ has Maclaurin series:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

You may assume that this series converges to $\log(1+x)$ for $|x| < 1$.

- (a) Write down the Maclaurin series for $\log(1+2t^2)$. For what values of t is your series valid?
- (b) Find a power series for $\int_0^x \log(1+2t^2) dt$, justifying your working by referring to an appropriate theorem. For what values of x is your series valid?

[5 marks]

12. (a) Find the degree 2 Taylor polynomial for $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$ about the point $x = 0$.
- (b) Use this to estimate $\cosh(\frac{1}{2})$.
- (c) Use the remainder estimate from Taylor's theorem to find a bound on the error of your estimate in (b). (You may assume that $e^{1/2} < 2$.)

[6 marks]

END OF EXAMINATION PAPER