Answers: Subspaces

1. U must be closed under addition and multiplication by scalars. That is:
   (1) if $\mathbf{x}, \mathbf{y} \in U$ then $\mathbf{x} + \mathbf{y} \in U$, and
   (2) if $\mathbf{x} \in U$ and $\alpha \in \mathbb{R}$ then $\alpha\mathbf{x} \in U$.

2. In $\mathbb{R}^3$, the subspaces of dimension 2 are the planes through the origin, and the subspaces of dimension 1 are the lines through the origin.

3. Let $\mathbf{x} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n$ and $\mathbf{y} = \beta_1\mathbf{v}_1 + \ldots + \beta_n\mathbf{v}_n$ be in the span $S = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ and let $\alpha \in \mathbb{R}$. Then
   $$\mathbf{x} + \mathbf{y} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \ldots + (\alpha_n + \beta_n)\mathbf{v}_n \in S$$
   and
   $$\alpha\mathbf{x} = (\alpha\alpha_1)\mathbf{v}_1 + \ldots + (\alpha\alpha_n)\mathbf{v}_n \in S.$$
   Also S is non-empty (for example $\mathbf{0} \in S$). Hence S is a subspace of $\mathbb{R}^4$.

   If $n \geq 4$, the maximum possible dimension for S is $\dim \mathbb{R}^4 = 4$.

4. True.

5. $\dim$ solution space = number of columns − rank $= n - n = 0$.

6. rank$A$ = dim(column space of$A$) = dim(row space of$A$)