**THE UNIVERSITY OF MELBOURNE**

**DEPARTMENT OF MATHEMATICS AND STATISTICS**

**SEMESTER 2, 2006**

# 620-143 APPLIED MATHEMATICS

*Exam duration — Three hours*

*Reading time — 15 minutes*

**This paper has 6 pages, including this cover sheet.**

**Examination Papers with Common Content:**

This paper contains some questions in common with those for the subjects 620–123 Applied Mathematics (Advanced) and 620–113 Applied Mathematics (Advanced Plus). Examinations for these subjects are being held at the same time.

**Instructions to Invigilators:**

Initially, students are to receive a 14 page script book.

**Authorized Materials:**

No calculators, computers or mobile phones are permitted.

No written or printed material may be brought into the examination room.

**Instructions to Students:**

There are 12 questions on this examination paper.

All questions may be answered.

Marks for each question are indicated on the paper.

The total number of marks on the exam paper is 150.

**Standard integrals and limits and some other formulae are on Page 6.**

1. Evaluate the following integrals:

(a) $\displaystyle\int_1^e \frac{1}{x}\,(\log x)^2\,dx$ ; (b) $\displaystyle\int \sqrt{9-x^2}\,dx$ ; (c) $\displaystyle\int \frac{7x^2+3}{x(x^2+1)}\,dx$ .

[17 marks]

2. Find the length of the curve $y = 2\cosh(\frac{x}{2})$ from $x = 0$ to $x = 10$.

[5 marks]

3. Consider the following integral

$$\int_0^1 \log x\,dx\,.$$

(a) Explain why the integral is improper.

(b) Evaluate the integral.

[8 marks]

4. Find the general solution of the following differential equation:

$$2x^2\,\frac{\mathrm{d}y}{\mathrm{d}x} = -10x^2 - y^2.$$

[9 marks]

5. An electric circuit contains a resistor of 1000 ohms and a capacitor of $10^{-4}$ farads connected in series to a voltage source

$$V(t) = 250\sin 5t.$$

Initially there is no charge on the capacitor.

(a) Show that the charge on the capacitor $q(t)$ satisfies the differential equation

$$\frac{dq}{dt} + 10q = \frac{1}{4}\sin 5t.$$

(b) Find the charge as a function of time.

(c) Identify in your solution the *steady-state* part and the *transient* part of the solution.

[13 marks]

6. (a) Briefly explain the difference between a sequence and a series and give an example of each.

   (b) For each of the following sequences, find the limit if it exists, or explain why it diverges. State which standard limits and rules you have used.

   i. $a_n = \dfrac{2n^3 - 5^n + 2^n}{5^n - 5}$;  ii. $b_n = \dfrac{n! + 3^n}{2^n + 3}$;

   iii. $c_n = \sqrt[n]{4 + n}$;  iv. $d_n = \dfrac{\log(n^2 + 3)}{\log(3n + 5)}$.

[18 marks]

7. (a) Do the following series converge or diverge? State clearly any tests you have used.

   i. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$,  ii. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$,  iii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$.

   (b) Consider the following series

$$\sum_{n=1}^{\infty} \log\left(\frac{n}{n+1}\right).$$

   i. Write out the first 3 partial sums and simplify them.
   ii. Deduce the $k$th partial sum of the series.
   iii. Use part (b) (ii) either to find the sum of the series if it exists or to explain why the series diverges.

[15 marks]

8. (a) Find the Taylor polynomial of degree 9 centred at $x = 0$ for the function

$$f(x) = \sinh x\ .$$

(b) Hence show that the Maclaurin series representing $\sinh x$ is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

(c) Find the radius of convergence of the series in part (b), showing your full working.

(d) Using the remainder estimate from Taylor's theorem, bound the error in approximating $\sinh 1$ by using the degree 9 Taylor polynomial about $x = 0$ for $\sinh x$.
You may use the facts: $\sinh 1 < \cosh 1 < 3$; $10! \approx 3.6 \times 10^6$.

(e) Using the series in part (b), find a power series that represents the function

$$\int \frac{\sinh x}{x}\, dx.$$

State the radius of convergence of this series, giving your reasons.

[15 marks]

9. (a) Using the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$, find the power series representing $\dfrac{1}{1+2x}$ and hence show that

$$\log|1+2x| = \sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}x^{n+1}}{n+1}.$$

in the interval of convergence of the series.

Show that the series can be rewritten as

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k x^k}{k}. \qquad (1)$$

(b) Find the radius of convergence of the series labeled (1) above.

(c) Find the interval of convergence of the series labeled (1) above.

[12 marks]

10. Consider the second order differential equation

$$y'' - 4y' + 4y = f(x).$$

(a) Find the general solution if $f(x) = 0$.

(b) Find a particular solution if $f(x) = 8x$.

(c) Find a particular solution if $f(x) = 25\cos x$.

(d) Find the general solution if $f(x) = 4x - 50\cos x$.

[15 marks]

11. Consider the third order differential equation

$$y''' + 3y'' - 4y' = f(x).$$

(a) Find the general solution if $f(x) = 0$.

(b) What function would you guess for the trial particular solution if $f(x)$ is

i. $e^x \sin x$, ii. $e^{-4x}$ ?

**Do NOT find the actual particular solutions.**

[6 marks]

12. A 10 kilogram mass suspended from the end of a vertical spring stretches the spring $\frac{49}{90}$ metres. At time $t = 0$, the mass is started in motion from the equilibrium position with an initial velocity of 1 m/s in the upward direction. At the same time, a constant downward force of 360 Newtons is applied to the system. Assume that air resistance is equal to 60 times the instantaneous velocity and that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.

(a) Determine the spring constant.

(b) Show that the equation of motion is

$$\ddot{x} + 6\dot{x} + 18x = 36$$

where $x$ is the displacement of the mass below the equilibrium position at time $t$. In your answer include a diagram of all forces acting on the mass.

(c) Find the position of the mass at any time.

[17 marks]

*END OF EXAMINATION*

*Standard integrals, limits and other formulae are on Page 6*

# Standard Integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \operatorname{arccosh}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \operatorname{arcsinh}\frac{x}{a} + C$$

where $a > 0$ is constant, $k$ is constant, and $C$ is an arbitrary constant of integration.

# Useful Formulae

**Basic Formulae**

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$

**Double Angle Formulae**

$\cos 2x = \cos^2 x - \sin^2 x$ $\qquad$ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$\sin 2x = 2\sin x\cos x$ $\qquad$ $\sinh 2x = 2\sinh x\cosh x$

# Standard limits

(i) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0)$

(ii) $\lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$

(iii) $\lim_{n\to\infty} a^{1/n} = 1 \quad (a > 0)$

(iv) $\lim_{n\to\infty} n^{1/n} = 1$

(v) $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (\text{all } a)$

(vi) $\lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$

(vii) $\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a \quad (\text{all } a)$

(viii) $\lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p > 0,\ a > 1)$