1. Consider the initial value problem:
\[ \frac{dy}{dx} = \log_e (y^2) - \sin \left( \frac{x}{y} \right) \quad \text{with } y = 2 \text{ when } x = 0. \]
Estimate \( y \) at \( x = 0.1 \) to 3 decimal places using a quadratic (2\text{nd} degree) Taylor polynomial about \( x = 0 \).
(4 marks)

2. Let \( f \) be defined by
\[ f(x) = \cos \left( \log_e (x) \right). \]
Derive the 3rd degree Taylor Polynomial for \( f(x) \) about \( x = 1 \). \textit{Express all coefficients as vulgar fractions, NOT decimals.}
(3 marks)

3. Consider the initial value problem:
\[ \frac{dy}{dx} = 2y\sqrt{x} \quad \text{with } y = 1 \text{ when } x = 0. \]
(a) State the algorithm for the Improved Euler method. Use the improved Euler method to compute \( y(x) \) when \( x = 0.2 \). Use \( \Delta x = 0.1 \) — that is, use two iterations. State any numbers 4 decimal places.
(b) Show that \( y = \exp \left( \frac{4}{3}x^{3/2} \right) \) is the solution to the above initial value problem, and so compute \( y(0.2) \) to 4 decimal places.
(6 marks)

4. Find the antiderivatives of the following functions:
(a) \[ \frac{x^2 - 3x + 8}{x^2 + 2x - 15} \]
(b) \[ \frac{2}{x} \sin \left( \log_e(x) \right) \]
(c) \( x^2 \sin(x) \) \textit{(hint: be persistent)}.  
(6 marks)

5. Find the solution of the following two initial value problems:
(a) \[ \frac{dy}{dx} = \frac{xy}{1 + x^2} \quad \text{with } y(0) = 2. \]
(b) \[ x^2 \frac{dy}{dx} + 3xy = \frac{1}{x}e^x \quad \text{with } y(1) = 1. \]
(6 marks)