THE UNIVERSITY OF MELBOURNE
DEPARTMENT of MATHEMATICS AND STATISTICS
620-151 INTRODUCTION TO BIOMEDICAL MATHEMATICS
MAIN EXAMINATION — Semester 1, 2004

Common content examinations — Nil
Exam duration — Three hours
Reading time — 15 minutes
This paper consists of 5 pages

- **Authorised Materials:** A scientific calculator may be used with this examination. No graphics or programmable calculators are allowed in this examination. No calculator capable of algebraic manipulations is allowed in this examination. If you have any such calculators in your possession, you should immediately give them to an invigilator.

  Pens, pencils, rubbers, rulers and a MATH-O-MATT may be used. Candidates may bring into the examination one A4 sheet of notes/formulae (doubled-sided) in their own handwriting. No other written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately give it to an invigilator.

- **Instructions to Invigilators:** Each candidate should be issued with a 12-page or 14-page script booklet. Further booklets (of appropriate size) should be issued on demand. The students may remove the exam paper at the conclusion to the examination.

- **Examination paper:** Candidates are permitted to take this question paper with them at the end of the examination.

- **Instructions to students:** This examination consists of **ten questions**. All questions may be answered and all questions are of equal weight. Students are required to show intermediate steps in the solution of the questions: answers alone are not sufficient.

- **Reproduction of question paper:** *This paper may be reproduced for library storage and related purposes.*

1. Use *strict* Gauss-Jordan elimination with augmented matrices on each of the systems of linear equations below to find all the solution(s), if one or more exist, for (x, y, z).

   In the case of a single solution, your final matrix should allow you to answer the question without further algebraic manipulation: your augmented matrix *must* be in Reduced Row Echelon form. If multiple solutions exist, state them all. If no solution exists, state why such a deduction can be made from your final matrix.

   (a)
$$\begin{array}{rcrcrcr} 3x & - & 15y & + & 21z & = & 54 \\ -2x & + & 8y & - & 12z & = & -32 \\ -x & + & 6y & - & 5z & = & -5 \end{array}$$

   (b)
$$\begin{array}{rcrcrcr} -3x & + & 15y & - & 48z & = & -48 \\ 4x & - & 22y & + & 70z & = & 70 \\ 2x & - & 4y & + & 14z & = & 16 \end{array}$$

   Show the row operations used at each step. Check your answers.

2. (a) Given the matrices

$$A = \begin{bmatrix} 7 & -8 \\ 3 & 5 \\ -9 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 & -5 & 5 \\ -7 & 1 & -1 & 3 \end{bmatrix},$$

   find the product AB.

   (b) Let U and V be the following two matrices:

$$U = \begin{bmatrix} 1 & -4 & -3 \\ -3 & -7 & -6 \\ 2 & 6 & 5 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 11 & 15 \\ -4 & -14 & -19 \end{bmatrix}.$$

   Given that $UV = VU = I$, where I is the 3×3 multiplicative identity matrix, solve the (x, y, z) in terms of α in the following system:

$$\begin{array}{rcrcrcr} 2x & - & 8y & - & 6z & = & -6\alpha \\ -6x & - & 14y & - & 12z & = & 4\alpha \\ 4x & + & 12y & + & 10z & = & 2\alpha \end{array}$$

   Do *not* use Gauss-Jordan elimination. Check your answer.

3. (a) Two variables, x and y, are related by the equation:

$$\frac{x^3y^2}{8} - \log_e(x+y) = 1.$$

When $x = 2$ and $y = -1$, x is increasing at a rate of 4 units per minute. What is the rate of change of y at $x = 2$ and $y = -1$?

(b) Find the slope of the curve

$$(x^2+y)e^{xy^2} - 3e^{-4} = 0$$

at $(x, y) = (-1, 2)$.

4. Find the antiderivative of the following functions:

(a) $f(x) = \dfrac{29-x}{x^2+5x-14}$

(b) $f(x) = \dfrac{3\cos(x) - 3\sin(x)}{\cos(x)+\sin(x)}$

(c) $f(x) = x\log_e x^\pi$

5. Use the simplex method to solve the following standard maximum problem.

$$\begin{array}{lrl} \text{Maximise} & h = 8x_1 + 5x_2 & +4x_3 \\ \text{subject to} & 3x_2 + 6x_3 & \leq 12 \\ & 2x_1 + x_2 + 4x_3 & \leq 6 \\ & 5x_1 + 2x_2 + 4x_3 & \leq 16 \end{array}$$

with $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

At each step show clearly the row operation(s) that you perform and clearly circle the pivot element. Inspect your final tableau and state the maximum possible value of h and all the values of (x_1, x_2, x_3) for which this maximum occurs. Check your answer.

6. (a) Find the Taylor Polynomial of degree 2 for the function

$$f(x) = \log_e\left(\frac{1}{\sqrt{x^2+e}}\right) \qquad \text{about the point } x = 0.$$

Express all coefficients as vulgar fractions, NOT decimals.

(b) (i) Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + 2\log_e(y) \qquad \text{with } y = 1 \text{ when } x = 1.$$

Find the Taylor Polynomial of degree 3 for $y(x)$ about $x = 1$.
Express all coefficients as vulgar fractions, NOT decimals.

(ii) Estimate the value of y at $x = 1.1$ to 3 decimal places using the Taylor polynomial above.

7. Solve graphically the following *non-standard* linear programming problem: Draw a graph with feasible region clearly marked and with all its corner points calculated. Write down all basic feasible solutions and write down the values of (x, y) for which P takes its maximum value.

$$\text{Maximise} \qquad P = 39x + 13y$$

$$\text{subject to} \qquad \begin{aligned} 21x + 7y &\le 63 \\ 12x - 6y &\ge -24 \\ 42x - 14y &\le 42 \end{aligned}$$

with $x \ge 0$ and $y \ge 0$. Check your answer.

8. (a) Find the general solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\cos(\pi x)}{y}$$

(b) Solve the initial value problem, for $t \ge 1$,

$$2t\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 12\sqrt{t^3}\,e^{t^3} \qquad \text{and} \qquad x(1) = 2e + 2$$

Show all intermediate steps in your solution of the differential equation.

9. A population of p million after t years is modelled by the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 9p - p^2 - 14$$

   (a) Find the equilibrium solutions and determine whether or not they are stable. Make sure that you give a clear argument for your answer.

   (b) Assume that the initial population is $p = 3.141$. Find the population after t years have elapsed. State what happens as t becomes very large. Show all intermediate steps in your solution of the differential equation. State any constants to 3 decimal places.

10. Five grams of a chemical Q are formed by combining 2 grams of a chemical M and 3 grams of a chemical L. Initially there are 80 grams of M, 300 grams of L and no chemical Q. The rate of formation of Q is proportional to the product of the unconverted amounts of M and L. It is observed that exactly 20 grams of Q have been formed after 5 minutes.

    (a) If $x(t)$ is the amount of chemical Q in grams at time t minutes, write down an equation for $\frac{\mathrm{d}x}{\mathrm{d}t}$ in terms of x and t.

    (b) Find $x(t)$ and hence deduce how much chemical Q is left after 20 minutes.

    (c) How much chemical M is left after a very long time?

    State decimal numbers to 2 decimal places.