7. (a) (i) Find the Taylor polynomial of degree 3 for the function $(2+2x)^{5/2}$ about the point $x = 1$.
Express all coefficients as vulgar fractions, NOT decimals.

(ii) Estimate y at $x = 1.10$ to 5 decimal places using the Taylor Polynomial calculated above.

(b) Consider the initial value problem

$$\frac{dy}{dx} = 2x^3 - 4y^2$$
$$\text{with} \quad y = 1.0 \quad \text{when} \quad x = 2.0$$

Find the Taylor Polynomial of degree 3 for $y(x)$ about $x = 2$.
Express all coefficients as vulgar fractions, NOT decimals.

7(a) (i) Let $f(x) = (2+2x)^{5/2}$

Hence

$$f'(x) = \frac{5}{2} \times 2 \times (2+2x)^{3/2}$$
$$= 5\,(2+2x)^{3/2}$$

so

$$f'(1) = 5\,(2+2)^{3/2}$$
$$= 5\;4^{3/2}$$
$$= 5 \cdot 2^3 = 5 \times 8 = 40$$

and

$$f''(x) = 5 \times \frac{3}{2} \times 2 \times (2+2x)^{1/2}$$
$$= 15\,(2+2x)^{1/2}$$

so

$$f''(1) = 15\;4^{1/2} = 15 \times 2 = 30$$

and

$$f^{(3)}(x) = 15 \times \frac{1}{2} \cdot 2 \cdot (2+2x)^{-\frac{1}{2}}$$
$$= \frac{15}{(2+2x)^{1/2}}$$

So $f^{(3)}(1) = \frac{15}{4^{1/2}} = \frac{15}{2}$

Since $f(x) = (2+2x)^{5/2}$ we have

$$f(1) = (2+2)^{5/2} = 4^{5/2} = 2^5 = 32$$

Now

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2} f''(1)(x-1)^2 + \frac{1}{6} f^{(3)}(1)(x-1)^3$$

So $P_3(x) = 32 + 40(x-1) + 15(x-1)^2 + \frac{5}{4}(x-1)^3$

(ii) Hence

$$f(1.10) \approx P_3(1.10) = 32 + 40(1.10-1) + 15(1.1-1)^2 + \frac{5}{4}(1.1-1)^3$$

$$= 32 + 40 \times 0.1 + 15 \times 0.01 + \frac{5}{4} \times 0.001$$

$$= 32 + 4 + 0.15 + 0.00125$$

$$= 36.15125$$

(b)

We have $y(2)=1$

and $y' = 2x^3 - 4y^2$

Hence $y'(2) = 2 \times 2^3 - 4 \times 1^2$

$= 16 - 4 = 12$

and $y'' = 6x^2 - 8yy'$

so $y''(2) = 6.2^2 - 8.1.12$

$= 24 - 96 = -72$

and $y^{(3)}(x) = 12x - 8(y')^2 - 8yy''$

so $y^{(3)}(2) = 24 - 8 \times 12^2 - 8 \times 1 \times (-72)$

$= 24 - 1152 + 576$

$= -552$

So $y(x) \approx P_3(x) = y(2) + y'(2)(x-2)$

$$+ \frac{y''(2)}{2!}(x-2)^2 + \frac{y^{(3)}(2)}{3!}(x-2)^3$$

So $P_3(x) = 1 + 12(x-2) - 36(x-2)^2$

$$- 92(x-2)^3$$

6. Find the antiderivative for each of the following:

(i) $$\frac{5x+13}{x^2+6x+5}$$

(ii) $$\frac{\log_e(x)}{x}$$

(iii) $$x\log_e(x)$$

(iv) $$x^2e^{5x^3+7}$$

6(i)

Let $\dfrac{5x+13}{(x+5)(x+1)} = \dfrac{A}{x+5} + \dfrac{B}{x+1}$

Hence

$$5x+13 = A(x+1) + B(x+5)$$

$\Rightarrow$ $5x+13 = (A+B)x + (A+5B)$ for all x.

Hence

$$A+B=5 \quad (\alpha)$$

and $A+5B=13 \quad (\beta)$

(α) implies $A = 5-B$ and substituting into (β) gives $5-B+5B=13$

so $4B = 8 \Rightarrow B=2$

and so $A=3$

Hence

$$\int \frac{5x+13}{(x+5)(x+1)}\,dx = \int \frac{3}{x+5}\,dx + \int \frac{2}{x+1}\,dx$$

$$= 3\log_e|x+5| + 2\log_e|x+1| + C$$

$$= \log_e\left|(x+5)^3(x+1)^2\right| + C$$

(ii) Consider $\int \frac{1}{x}\log_e x \, dx$

Let $u = \log_e x$ so $\frac{du}{dx} = \frac{1}{x}$

Hence

$$\int \frac{1}{x}\log_e x \, dx = \int \frac{1}{x}\log_e x \frac{1}{\left(\frac{du}{dx}\right)} \cdot du$$

$$= \int \frac{1}{x}\log_e x \frac{1}{(1/x)} du$$

$$= \int \log_e x \, du$$

$$= \int u \, du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{1}{2}(\log_e x)^2 + C$$

(iii) $\int x \log_e x \, dx$

Let $u = \log_e x$ and $\frac{dv}{dx} = x$ so that $v = \frac{x^2}{2}$

So $\int x \log_e x \, dx = \int u \frac{dv}{dx} \, dx$

$$= uv - \int v \frac{du}{dx} \, dx$$

$$= \frac{x^2}{2} \log_e x - \int \frac{x^2}{2} \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \log_e x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \log_e x - \frac{1}{2} \cdot \frac{x^2}{2} + C$$

$$= \frac{x^2}{4} (2 \log_e x - 1) + C$$

(iv) $$\int x^2 e^{5x^3+7}\,dx$$

Let $u = 5x^3+7$ so $\frac{du}{dx} = 15x^2$

So
$$\int x^2 e^{5x^3+7}\,dx = \int x^2 e^{5x^3+7}\frac{1}{15x^2}\,du$$

$$= \frac{1}{15}\int e^u\,du$$

$$= \frac{1}{15}e^u + C$$

$$= \frac{1}{15}e^{(5x^3+7)} + C$$