8. (a) Solve the initial value problem

$$\frac{dy}{dx} = -x^3y^5 \quad \text{and} \quad y(0) = -\frac{1}{2}$$

for $y(x)$. Show all intermediate steps in your solution of the differential equation.

(b) Find the general solution $x(t)$ of the differential equation

$$\frac{dx}{dt} - \frac{6}{t}x = 4t.$$

Show all intermediate steps in your solution of the differential equation.

8(a) We have

$$\frac{dy}{dx} = -x^3 y^5$$

$$y(0) = -\frac{1}{2}$$

Rewriting the DE gives

$$\frac{1}{y^5}\frac{dy}{dx} = -x^3$$

Integrating both sides with respect to x gives

$$\Rightarrow \int \frac{1}{y^5}\frac{dy}{dx} \cdot dx = -\int x^3 dx + C$$

$$\Rightarrow \int \frac{1}{y^5} dy = -\frac{x^4}{4} + C$$

$$\Rightarrow -\frac{1}{4} y^{-4} = -\frac{x^4}{4} + C$$

$$\Rightarrow y^{-4} = x^4 - 4C$$

Let $B = -4C$

So $$y^4 = \frac{1}{x^4+B}$$

$$\Rightarrow \quad y = \pm \frac{1}{\sqrt[4]{x^4+B}} \quad \text{or} \quad \pm \frac{1}{(x^4+B)^{1/4}}$$

Since $y(0) = -\frac{1}{2}$

$$-\frac{1}{2} = \pm \frac{1}{(0+B)^{1/4}}$$

Hence we must choose the negative branch of the 4th root and so

$$-\frac{1}{2} = -\frac{1}{B^{1/4}}$$

$$\Rightarrow \quad B^{1/4} = 2$$

$$\Rightarrow \quad B = 2^4 = 16$$

Hence the solution of the initial value problem is

$$y = \frac{-1}{(x^4+16)^{1/4}}$$

(b)
$$\frac{dx}{dt} - \frac{6}{t}x = 4t$$

An integrating factor $I(t)$ is

$$I(t) = e^{\int -\frac{6}{t}\,dt}$$
$$= e^{-6\log_e|t|}$$
$$= e^{\log_e|t^{-6}|}$$
$$= |t^{-6}|$$
$$= t^{-6} \quad (\text{since } (-t)^6 = t^6)$$

So multiplying both sides of the DE by t^{-6}

gives
$$t^{-6}\frac{dx}{dt} - 6t^{-7}x = 4t^{-5}$$

$$\Rightarrow \quad \frac{d(t^{-6}x)}{dt} = 4t^{-5}$$

$$\Rightarrow t^{-6}x = 4\int t^{-5}\,dt + C$$

$$\Rightarrow t^{-6}x = 4\left(-\frac{1}{4}t^{-4}\right) + C$$

$$\Rightarrow t^{-6}x = -t^{-4} + C$$

$$\Rightarrow x = -t^{2} + C\,t^{-6}, \quad C \in \mathbb{R}.$$

That is
$$x(t) = -t^{2} + \frac{C}{t^{6}}, \quad C \in \mathbb{R}$$

is the general solution of the DE.

10. A tank initially contains 50 litres of alcohol and 25 litres of water. Pure water is run into the tank at 3 litres/min and the mixture is continuously stirred. The mixture is withdrawn through a tap at the bottom of the tank at a constant rate of 2 litres/min.

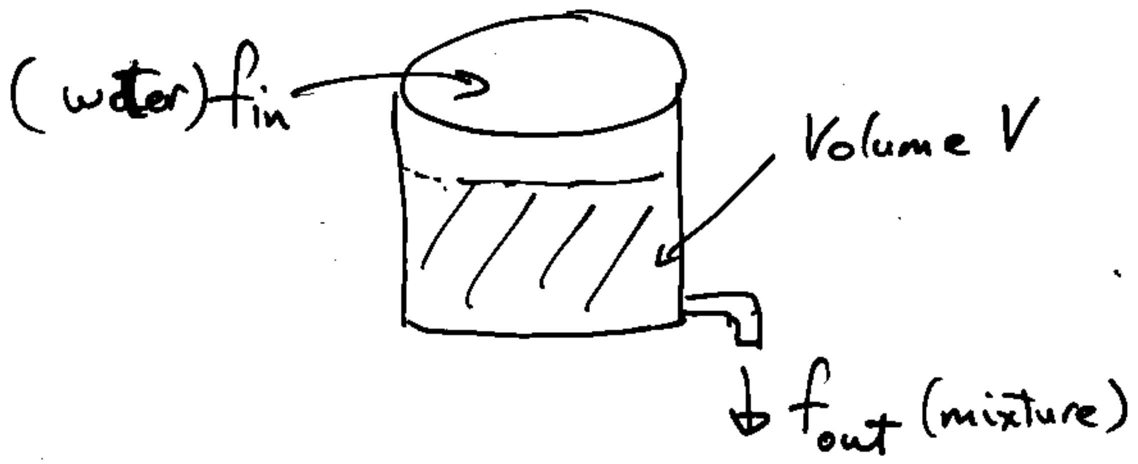
(a) What is the total volume of mixture in the tank after t minutes?

(b) Supposing the mixture contains x litres of water after t minutes, write down $\frac{dx}{dt}$ in terms of x and t.

(c) When will the mixture in the tank contain equal quantities of alcohol and water?

①

10. Let the volume of the mixture after t minutes be V litres. Let f_{in} be the rate (l/min) at which water is run into the tank and f_{out} be the rate (l/min) at which the mixture is withdrawn.



We are told that the tank initially contains 50 l of alcohol and 25 l of water and so has 75 l of mixture initially.

Hence
$$V(0) = 75$$

We have $f_{in} = 3$ and $f_{out} = 2$

(a) Since

$$V(t) = V(0) + (f_{in} - f_{out})t$$

So $V(t) = 75 + (3-2)t$

$$= 75 + t$$

(b) We are told that there is x litres of water after t minutes.

Let C be the proportion of water in the mixture. So

$$C = \frac{x}{V}$$

Since pure water is added to the tank

$$\frac{dx}{dt} = f_{in} - f_{out} C$$

So

$$\frac{dx}{dt} = f_{in} - f_{out}\,\frac{x}{V}$$

$$\Rightarrow \quad \frac{dx}{dt} = 3 - \frac{2x}{V}$$

$$\Rightarrow \quad \frac{dx}{dt} = 3 - \frac{2x}{(75+t)}$$

(c). The amount of alcohol in the mixture is $V-x$ litres so the point when the mixture contains equal amounts of water and alcohol is given by

$$x = V - x$$

$$\Rightarrow \quad x = \frac{V}{2}$$

Let $t = T$ be the time in minutes when this happens. So

$$x = \frac{75+T}{2}.$$

Now the IVP is

$$\left.\begin{aligned} \frac{dx}{dt} + \frac{2}{(75+t)}x &= 3 \\ x(0) &= 25 \end{aligned}\right\}$$

To find the general solution of the DE we find an integrating factor $I(t)$

$$I(t) = e^{\int \frac{2}{75+t}\,dt}$$

$$= e^{2\log_e|75+t|}$$

$$= |(75+t)^2|$$

$$= (75+t)^2$$

Multiplying both sides of the DE by $I(t)$ gives

$$(75+t)^2\frac{dx}{dt} + 2(75+t)x = 3(75+t)^2$$

$$\Rightarrow \quad \frac{d}{dt}\left((75+t)^2 x\right) = 3(75+t)^2$$

Integrating both sides with respect to t gives

$$(75+t)^2 x = 3\int (75+t)^2 dt + C$$

$$\Rightarrow \quad (75+t)^2 x = 3\frac{(75+t)^3}{3} + C$$

$$\Rightarrow \quad x = (75+t) + \frac{C}{(75+t)^2}$$

Since $x(0) = 25$

$$25 = 75 + \frac{C}{75^2}$$

$$\Rightarrow \quad \frac{c}{75^2} = -50$$

$$\Rightarrow \quad c = -50 \times 75^2$$

Hence the solution of the IVP above is

$$x(t) = 75 + t - 50\left(\frac{75}{75+t}\right)^2$$

We want the time $t = T$ such that

$$x = \frac{V}{2} \qquad (*)$$

Now $V = 75 + t$ and $x = x(t) = 75 + t - 50\left(\frac{75}{75+t}\right)^2$

so we have

$$75 + T - 50\left(\frac{75}{75+T}\right)^2 = \frac{75+T}{2} \qquad (\Delta)$$

at such a time.

Now equation (Δ) implies

$$1 - \frac{50 \times 75^2}{(75+T)^3} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{50 \times 75^2}{(75+T)^3} = \frac{1}{2}$$

$$\Rightarrow \quad 2 \times 50 \times 75^2 = (75+T)^3$$

$$\Rightarrow \quad (75+T) = (100 \times 75^2)^{1/3}$$

$$\Rightarrow \quad T = (100 \times 75^2)^{1/3} - 75$$

$$\cong 7.548 \text{ minutes}.$$

11. A population of x million after t years is modelled by the equation

$$\frac{dx}{dt} = 11x - x^2 - 28$$

(a) Find the equilibrium solutions and determine whether they are stable or unstable, making sure you clearly state the argument for your answer as well as your answer.

(b) Find the population after t years have elapsed if the initial population is $x = 4.590$, and state what happens as t becomes very large. Show all intermediate steps in your solution of the differential equation. State any constants to 3 decimal places.

(a)

Our D.E. is

$$\frac{dx}{dt} = 11x - x^2 - 28$$

$$\Rightarrow \quad \frac{dx}{dt} = -(x-7)(x-4)$$

The equilibrium solutions occur

when $\frac{dx}{dt} = 0$

that is, when $(x-7)(x-4) = 0$

Hence $x(t) = 4$ and $x(t) = 7$

are equilibrium solutions.

Either:

Since for $x < 4$ $\frac{dx}{dt} < 0$, while for $4 < x < 7$ $\frac{dx}{dt} > 0$, the equilibrium at $x = 4$ is <u>unstable</u> as any initial condition near but not equal to 4 leads to a solution that evolves away from $x = 4$.

For $x > 7$ $\frac{dx}{dt} < 0$, and for $4 < x < 7$ $\frac{dx}{dt} > 0$, so any initial condition near to but not equal to 7 leads to a solution that evolves towards $x = 7$. Hence $x = 7$ is a <u>stable</u> equilibrium.

OR. Let u be such that $x = 4 + u$

so $$\frac{d(4+u)}{dt} = -(-3+u)u \approx +3u$$

$$\Rightarrow \quad \frac{du}{dt} \approx 3u \qquad \text{for } |u| \ll 1$$

So $$u \approx u_0 e^{3t}$$

Hence if $|u_0| \ll 1$ $|u|$ gets large as $t \to \infty$ and so the solution moves away from $x = 4$. Hence $x = 4$ is an unstable equilibrium.

Let v be such that $x = 7 + v$

So $$\frac{d(7+v)}{dt} = -v(3+v)$$

$$\Rightarrow \quad \frac{dv}{dt} \approx -3v \qquad \text{so } v \approx v_0 e^{-3t}$$

for $|v| \ll 1$.

Hence if $|V_0| \ll 1$ $|V|$ gets even smaller ($|V| \ll |V_0|$ as $t \to \infty$) as $t \to \infty$ and so the solution moves towards $x = 7$

Hence $x = 7$ is a stable equilibrium

(b) We have the IVP

$$\left.\begin{aligned} \frac{dx}{dt} &= (x-4)(7-x) \\ x(0) &= 4\cdot 59 \end{aligned}\right\}$$

The DE is

$$\frac{1}{(x-4)(7-x)}\frac{dx}{dt} = 1$$

$$\Rightarrow \int \frac{1}{(x-4)(7-x)}\frac{dx}{dt}\,dt = \int 1\,dt + C$$

$$\Rightarrow \int \frac{1}{(x-4)(7-x)}\,dx = \int 1\,dt + c$$

Now we can let

$$\frac{1}{(x-4)(7-x)} = \frac{A}{(x-4)} + \frac{B}{7-x}$$

So

$$1 = A(7-x) + B(x-4)$$

$$\Rightarrow \quad 1 = (7A - 4B) + (B-A)x$$

So $\quad 7A - 4B = 1 \qquad (\alpha)$

and $\quad B - A = 0 \qquad (\beta)$

Equation (β) implies $B = A$ and substituting into (α) gives

$$7A - 4A = 1$$

$$\Rightarrow \quad 3A = 1 \quad \Rightarrow \quad A = \frac{1}{3}$$

So $\quad A = B = \frac{1}{3}$

Hence we have

$$\frac{1}{3}\int \frac{1}{x-4}\,dx + \frac{1}{3}\int \frac{1}{7-x}\,dx = t + C$$

$$\Rightarrow \log_e|x-4| - \log_e|7-x| = 3t + 3C$$

$$\Rightarrow \log_e \left| \frac{x-4}{7-x} \right| = e^{3t} e^{3c}$$

$$\Rightarrow \frac{(x-4)}{(7-x)} = \pm e^{3c} \, e^{3t}$$

Let $A = \pm e^{3c}$ so that $-\infty < A < \infty$.

Now since $x(0) = 4.590$ we have

$$\frac{4.59 - 4}{7 - 4.59} = A e^{0}$$

So $A = \dfrac{0.590}{2.410} \approx 0.245$

Hence

$$\frac{x-4}{7-x} = 0.245 e^{3t}$$

$$\Rightarrow x - 4 = 0.245 e^{3t} (7 - x)$$

$$\Rightarrow x \left(1 + 0.245 e^{3t}\right) = 7 \times 0.245 e^{3t} + 4$$

Hence
$$x = \frac{4 + 1.715e^{3t}}{1 + 0.245e^{3t}}$$

is the solution of the IVP.

As $t \to \infty$ we known that $e^{3t} \to \infty$ while $e^{-3t} \to 0$ so rewriting our solution as

$$x = \frac{4e^{-3t} + 1.715}{e^{-3t} + 0.245}$$

allows us to deduce that as $t \to \infty$

$$x \to \frac{1.715}{0.245} = 7$$

That is, the population tends towards 7 million individuals.