THE UNIVERSITY OF MELBOURNE
DEPARTMENT of MATHEMATICS AND STATISTICS
620-151 INTRODUCTION TO BIOMEDICAL MATHEMATICS
EXAMINATION — Semester 1, 2004

Solutions.

1. Use *strict* Gauss-Jordan elimination with augmented matrices on each of the systems of linear equations below to find all the solution(s), if one or more exist, for (x_1, x_2, x_3).

   In the case of a single solution, your final matrix should allow you to answer the question without further algebraic manipulation: your augmented matrix *must* be in Reduced Row Echelon form. If multiple solutions exist, state them all. If no solution exists, state why such a deduction can be made from your final matrix.

   (a) $$\begin{array}{rcrcrcr} 3x & - & 15y & + & 21z & = & 54 \\ -2x & + & 8y & - & 12z & = & -32 \\ -x & + & 6y & - & 5z & = & -5 \end{array}$$

   (b) $$\begin{array}{rcrcrcr} -3x & + & 15y & - & 48z & = & -48 \\ 4x & - & 22y & + & 70z & = & 70 \\ 2x & - & 4y & + & 14z & = & 16 \end{array}$$

   Show the row operations used at each step. Check your answers.

   (a) Write as augmented matrix

   $$\left[\begin{array}{ccc|c} 3 & -15 & 21 & 54 \\ -2 & 8 & -12 & -32 \\ -1 & 6 & -5 & -5 \end{array}\right]$$

   Divide R1 by 3

   $$\left[\begin{array}{ccc|c} 1 & -5 & 7 & 18 \\ -2 & 8 & -12 & -32 \\ -1 & 6 & -5 & -5 \end{array}\right]$$

   Add 2*R1 to R2. Add R1 to R3

   $$\left[\begin{array}{ccc|c} 1 & -5 & 7 & 18 \\ 0 & -2 & 2 & 4 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

   Divide R2 by -2

   $$\left[\begin{array}{ccc|c} 1 & -5 & 7 & 18 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

   Add 5*R2 to R1. Add -1*R2 to R3

   $$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 3 & 15 \end{array}\right]$$

   Divide R3 by 3

   $$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Add R3 to R2. Add -2*R3 to R1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Hence $(x_1, x_2, x_3) = (-2, 3, 5)$. Check the answer:

$$\begin{array}{ccccc} 3 \times -2 & -15 \times 3 & +21 \times 5 & = -6 - 45 + 105 & = 54 \\ -2 \times -2 & +8 \times 3 & -12 \times 5 & = 4 + 24 - 60 & = -32 \\ -1 \times -2 & +6 \times 3 & -5 \times 5 & = 2 + 18 - 25 & = -5 \end{array}$$

- Write as augmented matrix

$$\left[\begin{array}{ccc|c} -3 & 15 & -48 & -48 \\ 4 & -22 & 70 & 70 \\ 2 & -4 & 14 & 16 \end{array}\right]$$

Divide R1 by -3

$$\left[\begin{array}{ccc|c} 1 & -5 & 16 & 16 \\ 4 & -22 & 70 & 70 \\ 2 & -4 & 14 & 16 \end{array}\right]$$

Add -4*R1 to R2. Add -2*R1 to R3.

$$\left[\begin{array}{ccc|c} 1 & -5 & 16 & 16 \\ 0 & -2 & 6 & 6 \\ 0 & 6 & -18 & -16 \end{array}\right]$$

Divide R2 by -2

$$\left[\begin{array}{ccc|c} 1 & -5 & 16 & 16 \\ 0 & 1 & -3 & -3 \\ 0 & 6 & -18 & -16 \end{array}\right]$$

Add 5*R2 to R1. Add -6*R2 to R3

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

STOP! We see that the last row is equivalent to the equation $0 = 2$ which does not make sense. Hence the system of equations is inconsistent and does not have a solution.

2. (a) Given the matrices

$$A = \begin{bmatrix} 7 & -8 \\ 3 & 5 \\ -9 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & -1 & -5 & 5 \\ -7 & 1 & -1 & 3 \end{bmatrix},$$

find the product AB.

(b) Let U and V be the following two matrices:

$$U = \begin{bmatrix} 1 & -4 & -3 \\ -3 & -7 & -6 \\ 2 & 6 & 5 \end{bmatrix} \qquad \text{and} \qquad V = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 11 & 15 \\ -4 & -14 & -19 \end{bmatrix}.$$

Given that $UV = VU = I$, where I is the 3×3 multiplicative identity matrix, solve the (x, y, z) in terms of α in the following system:

$$\begin{array}{rcrcrcr} 2x & - & 8y & - & 6z & = & -6\alpha \\ -6x & - & 14y & - & 12z & = & 4\alpha \\ 4x & + & 12y & + & 10z & = & 2\alpha \end{array}$$

Do *not* use Gauss-Jordan elimination. Check your answer.

(a) Do very explicit matrix multiplication. A is a 3×2 matrix and B is a 2×4 matrix. Hence the product AB will be a 3×4 matrix:

$$AB = \begin{bmatrix} 7 \times 0 + -8 \times -7 & 7 \times -1 + -8 \times 1 & 7 \times -5 + -8 \times -1 & 7 \times 5 + -8 \times 3 \\ 3 \times 0 + 5 \times -7 & 3 \times -1 + 5 \times 1 & 3 \times -5 + 5 \times -1 & 3 \times 5 + 5 \times 3 \\ -9 \times 0 + 4 \times -7 & -9 \times -1 + 4 \times 1 & -9 \times -5 + 4 \times -1 & -9 \times 5 + 4 \times 3 \end{bmatrix}$$

which is

$$AB = \begin{bmatrix} 56 & -15 & -27 & 11 \\ -35 & 2 & -20 & 30 \\ -28 & 13 & 41 & -33 \end{bmatrix}$$

(b) Write the system as a matrix product

$$\begin{bmatrix} 2 & -8 & -6 \\ -6 & -14 & -12 \\ 4 & 12 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6\alpha \\ 4\alpha \\ 2\alpha \end{bmatrix}$$

We see that the matrix on the left-hand side is simply $2U$. Hence we can simplify the expression a bit:

$$2U \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$$

Since $VU = I$ we multiply both sides of the equation on the left by V:

$$2VU \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha V \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$$

This gives:

$$2I\begin{bmatrix}x\\y\\z\end{bmatrix} = 2\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha V\begin{bmatrix}-6\\4\\2\end{bmatrix}$$

or

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha V\begin{bmatrix}-3\\2\\1\end{bmatrix}$$

Multiplying out the right-hand side gives:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha\begin{bmatrix}4\\28\\-35\end{bmatrix}$$

Hence $(x, y, z) = (4\alpha, 28\alpha, -35\alpha)$. Checking the answer to be sure:

$$\begin{aligned}
2 \times 4\alpha - 8 \times 28\alpha - 6 \times -35\alpha &= \alpha(8 - 224 + 210) &= 6\alpha \\
-6 \times 4\alpha - 14 \times 28\alpha - 12 \times -35\alpha &= \alpha(-24 - 392 + 420) &= 4\alpha \\
4 \times 4\alpha + 12 \times 28\alpha + 10 \times -35\alpha &= \alpha(16 - 336 - 350) &= 2\alpha
\end{aligned}$$

3. (a) Two variables, x and y, are related by the equation:

$$\frac{x^3y^2}{8} - \log_e(x+y) = 1$$

When $x = 2$ and $y = -1$, x is increasing at a rate of 4 units per minute. What is the rate of change of y at $x = 2$ and $y = -1$?

(b) Find the slope of the curve

$$(x^2+y)e^{xy^2} - 3e^{-4} = 0$$

at $(x, y) = (-1, 2)$.

(a) Since we need to find $\frac{\mathrm{d}y}{\mathrm{d}t}$ and we are given $\frac{\mathrm{d}x}{\mathrm{d}t}$ we differentiate both sides of the equation with respect to t:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^3y^2}{8}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\log(x+y) &= \frac{\mathrm{d}}{\mathrm{d}t}1 \\
\frac{x^3}{8}\frac{\mathrm{d}}{\mathrm{d}t}y^2 + \frac{y^2}{8}\frac{\mathrm{d}}{\mathrm{d}t}x^3 - \frac{1}{x+y}\frac{\mathrm{d}}{\mathrm{d}t}(x+y) &= 0 \\
\frac{x^3}{8}2y\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y^2}{8}3x^2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{x+y}\left(\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}\right) &= 0 \\
\frac{x^3y}{4}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{3x^2y^2}{8}\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{x+y}\left(\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}\right) &= 0
\end{aligned}$$

We don't need to isolate $\frac{\mathrm{d}y}{\mathrm{d}t}$ yet. Substitute $x = 2, y = -1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ into the above:

$$\begin{aligned}
\frac{-8}{4}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{12}{8}4 - \frac{1}{1}\left(4 + \frac{\mathrm{d}y}{\mathrm{d}t}\right) &= 0 \\
-2\frac{\mathrm{d}y}{\mathrm{d}t} + 6 - 4 - \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}$$

Now isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2/3$$

(b) We need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, so we differentiate both sides of the equation with respect to x.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2+y)e^{xy^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}(3e^{-4}) &= 0 \\
(x^2+y)\frac{\mathrm{d}}{\mathrm{d}x}(e^{xy^2}) + e^{xy^2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y) + 0 &= 0 \\
(x^2+y)e^{xy^2}\frac{\mathrm{d}}{\mathrm{d}x}(xy^2) + e^{xy^2}\left(2x + \frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\
(x^2+y)\left(x\frac{\mathrm{d}}{\mathrm{d}x}y^2 + y^2\right) + 2x + \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(x^2+y)\left(2xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2\right) + 2x + \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

Do not isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$ yet. Substitute in $x = -1$ and $y = 2$ into the above and then isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}(1+2)\left(-4\frac{\mathrm{d}y}{\mathrm{d}x}+4\right)-2+\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ -12\frac{\mathrm{d}y}{\mathrm{d}x}+12-2+\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 10/11$ when $(x, y) = (-1, 2)$.

4. Find the antiderivative of the following functions:

   (a) $f(x) = \dfrac{29 - x}{x^2 + 5x - 14}$

   (b) $f(x) = \dfrac{3\cos(x) - 3\sin(x)}{\cos(x) + \sin(x)}$

   (c) $f(x) = x \log_e x^\pi$

   (a) This looks like a partial fractions question. Factorise as much as possible:
   $$\frac{29 - x}{x^2 + 5 - 14} = \frac{29 - x}{(x-2)(x+7)}$$
   Convert it to partial fraction form:
   $$\frac{29 - x}{(x-2)(x+7)} = \frac{A}{x-2} + \frac{B}{x+7} = \frac{A(x+7) + B(x-2)}{(x-2)(x+7)}$$
   Look at the numerators: $29 - x = A(x+7) + B(x-2)$. When $x = -7$ this gives $36 = -9B$ and so $B = -4$. When $x = 2$ this gives $27 = 9A$ and so $A = 3$. Hence
   $$\frac{29 - x}{(x-2)(x+7)} = \frac{3}{x-2} - \frac{4}{x+7}$$
   Check that we have the correct form:
   $$\frac{29 - x}{(x-2)(x+7)} = \frac{3}{x-2} - \frac{4}{x+7} = \frac{3(x+7) - 4(x-2)}{(x-2)(x+7)} = \frac{3x + 21 - 4x + 8}{x^2 + 5x - 14}.$$
   Now integrate:
   $$\int \frac{29 - x}{x^2 + 5 - 14}\mathrm{d}x = \int \frac{3\mathrm{d}x}{x-2} - \int \frac{4\mathrm{d}x}{x+7}$$
   The integral $\int \frac{\mathrm{d}x}{x+a} = \log_e |x + a|$ and so
   $$\int \frac{29 - x}{x^2 + 5 - 14}\mathrm{d}x = 3\log_e |x - 2| - 4\log_e |x + 7| + C$$
   where C is a constant. Simplifying a bit further gives:
   $$\int \frac{29 - x}{x^2 + 5 - 14}\mathrm{d}x = \log_e \left| \frac{(x-2)^3}{(x+7)^4} \right| + C$$

   (b) We see that the numerator is (almost) the derivative of the denominator. Hence we substitute $u = \cos(x) + \sin(x)$, and then $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos(x) - \sin(x)$. Thus
   $$\begin{aligned} \int \frac{3\cos(x) - 3\sin(x)}{\cos(x) + \sin(x)}\mathrm{d}x &= \int \frac{3}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\ &= \int \frac{3}{u}\mathrm{d}u \\ &= 3\log_e |u| + C \\ &= 3\log_e |\cos(x) + \sin(x)| + C \end{aligned}$$
   where C is some constant.

(c) First simplify:

$$f(x) = x \log_e x^\pi = \pi x \log_e x$$

This doesn't look like a partial fraction or a simple substitution, so it is probably an integration by parts.

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$$

Try with $u = \log_e x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x$. This means that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1/x \qquad \text{and } v = x^2/2.$$

Substitute this into the integrate-by-parts formula:

$$\begin{aligned} \int x \log_e x &= \frac{x^2}{2} \log_e x - \int \frac{x^2}{2} \frac{1}{x} \mathrm{d}x \\ &= \frac{x^2}{2} \log_e x - \int \frac{x}{2} \mathrm{d}x \\ &= \frac{x^2}{2} \log_e x - \frac{x^2}{4} + C \end{aligned}$$

Hence

$$\int x \log_e x^\pi = \frac{\pi x^2}{4} (2 \log_e x - 1) + C$$

where C is some constant.

5. Use the simplex method to solve the following standard maximum problem.

$$\text{Maximise} \quad h = 8x_1 + 5x_2 + 4x_3$$

$$\text{subject to} \quad \begin{aligned} 3x_2 + 6x_3 &\le 12 \\ 2x_1 + x_2 + 4x_3 &\le 6 \\ 5x_1 + 2x_2 + 4x_3 &\le 16 \end{aligned}$$

with $x_1 \ge 0$, $x_2 \ge 0$ and $x_3 \ge 0$.

At each step show clearly the row operation(s) that you perform and clearly circle the pivot element. Inspect your final tableau and state the maximum possible value of h and all the values of (x_1, x_2, x_3) for which this maximum occurs. Check your answer.

First add slack variables, s_1, s_2, s_3:

$$\begin{aligned} 3x_2 + 6x_3 + s_1 &= 12 \\ 2x_1 + x_2 + 4x_3 + s_2 &= 6 \\ 5x_1 + 2x_2 + 4x_3 + s_3 &= 16 \end{aligned}$$

with the constraint $s_1, s_2, s_3 \ge 0$.

Form the simplex tableau:

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS
0	3	6	1	0	0	0	12
2	1	4	0	1	0	0	6
5	2	4	0	0	1	0	16
−8	−5	−4	0	0	0	1	0

−8 is the most negative element of the bottom row, so the first column is the pivot column. Work out the ratios:

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS	*Ratio*
0	3	6	1	0	0	0	12	○
(2)	1	4	0	1	0	0	6	3
5	2	4	0	0	1	0	16	$\frac{16}{5}$
−8	−5	−4	0	0	0	1	0	

Hence the second row is the pivot row and “2” is the pivot element. Divide Row 2 by 2:

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS
0	3	6	1	0	0	0	12
1	1/2	2	0	1/2	0	0	3
5	2	4	0	0	1	0	16
−8	−5	−4	0	0	1	0	

Add -5*R2 to R3. Add 8*R2 to R4.

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS
0	3	6	1	0	0	0	12
1	1/2	2	0	1/2	0	0	3
0	−1/2	−6	0	−5/2	1	0	1
0	−1	12	0	4	0	1	24

We have only one choice of pivot column — the second column. Work out the ratios:

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS	$Ratio$
0	(3)	6	1	0	0	0	12	4
1	1/2	2	0	1/2	0	0	3	6
0	−1/2	−6	0	−5/2	1	0	1	◦
0	−1	12	0	4	0	1	24	

Hence the first row is the pivot row and "3" is the pivot element. Divide R1 by 3:

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS
0	1	2	1/3	0	0	0	4
1	1/2	2	0	1/2	0	0	3
0	−1/2	−6	0	−5/2	1	0	1
0	−1	12	0	4	0	1	24

Add -1/2*R1 to R2. Add 1/2*R1 to R3. Add 1*R1 to R4.

x_1	x_2	x_3	s_1	s_2	s_3	P	RHS
0	1	2	1/3	0	0	0	4
1	0	1	−1/6	1/2	0	0	1
0	0	−5	1/6	−5/2	1	0	3
0	0	14	1/3	4	0	1	28

Here we must stop because there are no negative elements in the bottom row. We can now read off the solution.

$$P = 28 - 14x_3 - s_1/3 - 4s_2$$

Hence the maximum P is 28 and $x_3 = s_1 = s_2 = 0$. Reading off the other variables we see that $x_1 = 1$ and $x_2 = 4$ and $s_3 = 3$. ie the maximum P is 28 occurring at $(x_1, x_2, x_3) = (1, 4, 0)$ and $(s_1, s_2, s_3) = (0, 0, 3)$.

Checking our answer:

$$P = 8 \times 1 + 4 \times 5 = 28$$

and the inequalities:

$$\begin{aligned} 3 \times 4 + 6 \times 0 &= 12 \\ 2 \times 1 + 1 \times 4 + 4 \times 0 &= 6 \\ 5 \times 1 + 2 \times 4 + 4 \times 0 &= 13 \end{aligned}$$

This also agrees with the slack variables.

6. (a) Find the Taylor Polynomial of degree 2 for the function

$$f(x) = \log_e \left(\frac{1}{\sqrt{x^2 + e}} \right) \qquad \text{about the point } x = 0.$$

Express all coefficients as vulgar fractions, NOT decimals.

(b) (i) Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + 2\log_e(y) \qquad \text{with } y = 1 \text{ when } x = 1.$$

Find the Taylor Polynomial of degree 3 for $y(x)$ about $x = 1$.
Express all coefficients as vulgar fractions, NOT decimals.

(ii) Estimate the value of y at $x = 1.1$ to 3 decimal places using the Taylor polynomial above.

(a) First simplify the function:

$$f(x) = -\frac{1}{2}\log_e(x^2 + e)$$

In order to find the Taylor polynomial of degree 2 we need to find f' and f'':

$$\begin{aligned} f'(x) &= -\frac{1}{2}\frac{1}{x^2+e}\frac{\mathrm{d}}{\mathrm{d}x}(x^2+e) = -\frac{x}{x^2+e} \\ f''(x) &= -\frac{(x^2+e)\frac{\mathrm{d}x}{\mathrm{d}x} - x\frac{\mathrm{d}}{\mathrm{d}x}(x^2+e)}{(x^2+e)^2} \\ &= -\frac{x^2+e-2x^2}{(x^2+e)^2} = \frac{x^2-e}{(x^2-e)^2} \end{aligned}$$

Hence at $x = 0$ we have

$$\begin{aligned} f(0) &= -1/2 \\ f'(0) &= 0 \\ f''(0) &= -1/e \end{aligned}$$

Substitute this into the Taylor polynomial formula:

$$\begin{aligned} P(x+a) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ P(a) &= -1/2 + 0 - \frac{1}{2e}a^2 \end{aligned}$$

And so our degree 2 Taylor polynomial approximation to the function $f(x)$ about $x = 0$ is

$$-1/2 - \frac{1}{2e}x^2$$

(b-i) In order to compute the Taylor polynomial of degree 3 about $x = 1$ we need to know $y(1), y'(1), y''(1)$ and $y'''(1)$. From the DE we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + 2\log_e y$$

and so setting $x = 1$ and $y = 1$ we get

$$y'(1) = 1 + 2\log 1 = 1$$

Differentiating both sides of the DE by x gives:

$$y''(x) = 2xy(x)y'(x) + y(x)^2 + \frac{2y'(x)}{y(x)}$$

Substituting $x = 1$, $y(1) = 1$ and $y'(1) = 1$ gives

$$y''(1) = 2 + 1 + 2 = 5$$

Differentiate again:

$$\begin{aligned} y''' &= 2yy' + 2x(y')^2 + 2xyy'' + 2yy' + 2\frac{y''y - (y')^2}{y^2} \\ &= 4yy' + 2xyy'' + 2x(y')^2 + 2\frac{y''y - (y')^2}{y^2} \end{aligned}$$

Substitute $x = 1, y(1) = 1, y'(1) = 1, y''(1) = 5$ into this:

$$y'''(1) = 4 + 10 + 2 + 2\frac{5-1}{1} = 24$$

We can now form the Taylor polynomial about $x = 1$:

$$\begin{aligned} P(x) &= y(1) + y'(1)(x-1) + \frac{1}{2}y''(1)(x-1)^2 + \frac{1}{6}y'''(1)(x-1)^3 \\ &= 1 + (x-1) + \frac{5}{2}(x-1)^2 + 4(x-1)^3 \end{aligned}$$

(b-ii) Using the above Taylor polynomial we can put $x = 1.1$ and get an approximation of $y(1.1)$:

$$\begin{aligned} y(1.1) &= 1 + 0.1 + \frac{5}{2} \times 0.01 + 4 \times 0.001 \\ &= 1 + 0.1 + 0.025 + 0.004 = 1.129 \end{aligned}$$

7. Solve graphically the following *non-standard* linear programming problem: Draw a graph with feasible region clearly marked and with all its corner points calculated. Write down all basic feasible solutions and write down the values of (x, y) for which P takes its maximum value.

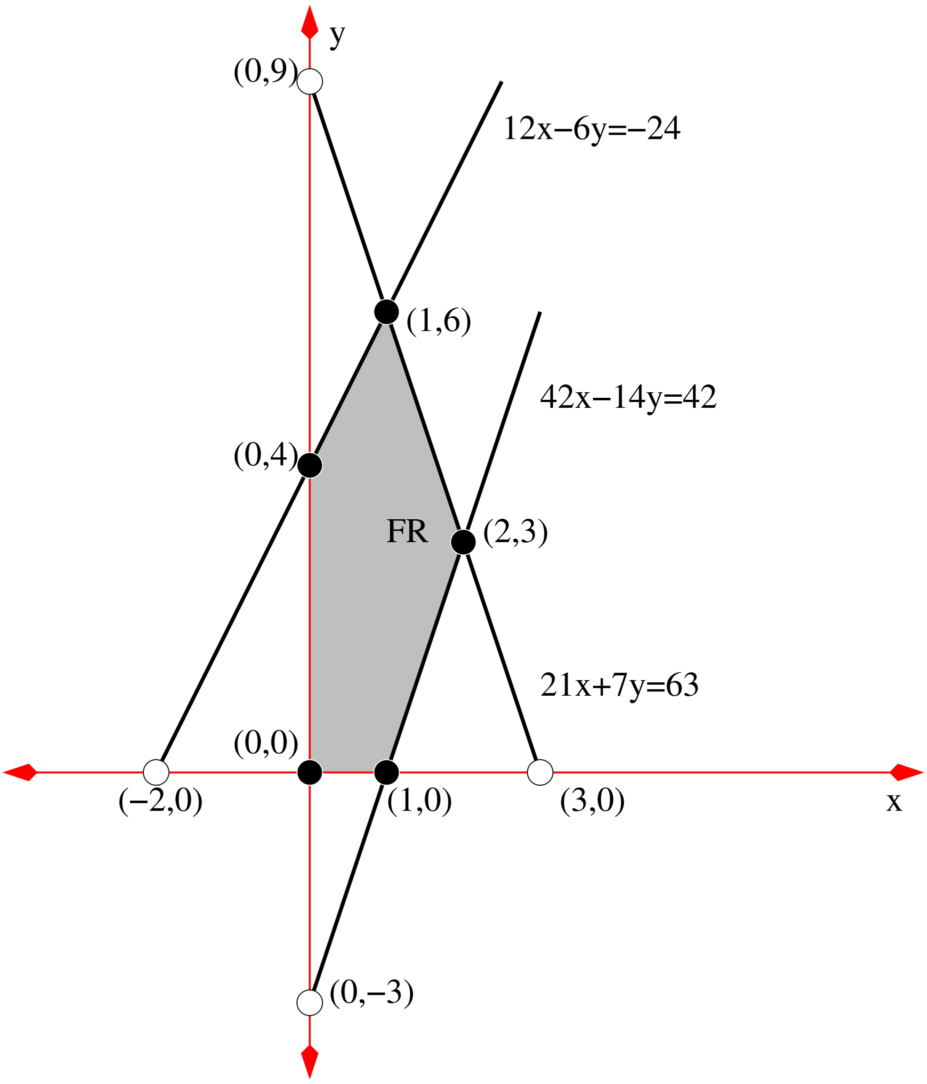
$$\begin{aligned} \text{Maximise} \qquad & P = 39x + 13y \\ \text{subject to} \qquad & 21x + 7y \leq 63 \\ & 12x - 6y \geq -24 \\ & 42x - 14y \leq 42 \end{aligned}$$

with $x \geq 0$ and $y \geq 0$. Check your answer.

Work out the intersection to create plot:

- Intersection of $x = 0$ and $y = 0$ is $(0, 0)$.
- Intersection of $y = 0$ and the constraints gives $(3, 0), (-2, 0)$ and $(1, 0)$.
- Intersection of $x = 0$ and the constraints gives $(0, 9)$, $(0, 4)$ and $(0, -3)$.

Use these to plot the graph:



Work out the remaining corner points by finding the intersections of the lines:

- Intersection of $21x + 7y = 63$ and $12x - 6y = -24$.

$$\left[\begin{array}{cc|c} -12 & 6 & 24 \\ 21 & 7 & 63 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & -4 \\ 3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & 0 & 5 \\ 3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 6 \end{array}\right]$$

  So the intersection is at $(x, y) = (1, 6)$.

- Intersection of $21x + 7y = 63$ and $42x - 14y = 42$.

$$\left[\begin{array}{cc|c} 21 & 7 & 63 \\ 42 & -14 & 42 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 1 & 9 \\ 3 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & 0 & 12 \\ 3 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & -1 & -3 \end{array}\right]$$

  So the intersection is at $(x, y) = (2, 3)$.

- Intersection of $12x - 6y = -24$ and $42x - 14y = 42$.

$$\left[\begin{array}{cc|c} 12 & -6 & -24 \\ 42 & -14 & 42 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & -4 \\ 3 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & -4 \\ 1 & 0 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -1 & -18 \\ 1 & 0 & 7 \end{array}\right]$$

  So the intersection is at $(7, 18)$.

Of these intersection points, only the following are basic feasible solutions:

- $(0, 0)$ — $P = 0$
- $(1, 0)$ — $P = 39$
- $(2, 3)$ — $P = 2 \times 39 + 3 \times 13 = 78 + 39 = 117$
- $(1, 6)$ — $P = 39 + 6 \times 13 = 39 + 78 = 117$
- $(0, 4)$ — $P = 4 \times 13 = 52$

Hence the maximum value of P is 117 and occurs at two different basic feasible solutions $(2, 3)$ and $(1, 6)$, and so also occurs at all the points on the line segment between then.

The equation for this line is $y = mx + c$. The gradient is $(6 - 3)/(1 - 2) = -3$. Hence $y = c - 3x$. It goes through the point $(1, 6)$, so $6 = c - 3$ and $c = 9$. The equation is $y + 3x = 9$.

Thus the maximum is

$$P = 117 \qquad \text{at all points } (x, 9 - 3x) \text{ where } 1 \leq x \leq 2$$

Check the answer.

- $P = 39x + 13y = 39x + 13(9 - 3x) = 13 \times 9 = 117$.
- $21x + 7y = 21x + 7(9 - 3x) = 63 \leq 63$.
- $12x - 6y = 12x - 6(9 - 3x) = -54 + 30x \geq -24$ provided $x \geq 1$.
- $42x - 14y = 42x - 14(9 - 3x) = 84x - 126 \leq 42$ provided $x \leq 2$.

8. (a) Find the general solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\cos(\pi x)}{y}$$

(b) Solve the initial value problem, for $t \geq 1$,

$$2t\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 12\sqrt{t^3}\, e^{t^3} \qquad \text{and} \qquad x(1) = 2e + 2$$

Show all intermediate steps in your solution of the differential equation.

(a) This DE is separable:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \times x\cos(\pi x)$$

Hence

$$\int x\cos\pi x \mathrm{d}x = \int y\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int y\mathrm{d}y$$

Solve the RHS first.

$$\int y\mathrm{d}y = \frac{y^2}{2} + C$$

where C is some constant.
Now solve the LHS. This is an integration by parts question.
Put $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos\pi x$. Hence $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \frac{1}{\pi}\sin\pi x$. Put this into the integration by parts formula:

$$\begin{aligned}
\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x &= uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\
\int x\cos\pi x\mathrm{d}x &= \frac{x\sin\pi x}{\pi} - \int \frac{\sin\pi x}{\pi}\mathrm{d}x \\
&= \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2} + D
\end{aligned}$$

where D is some constant.
Therefore we have

$$y^2 = \frac{2x\sin\pi x}{\pi} + \frac{2\cos\pi x}{\pi^2} + B$$

where $B = 2(C + D)$ is some constant. Solve for y:

$$y = \pm\sqrt{\frac{2x\sin\pi x}{\pi} + \frac{2\cos\pi x}{\pi^2} + B}$$

(b) We solve the DE using an integrating factor. First put it in standard form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3}{2t}x = 6\sqrt{t}\, e^{t^3}$$

Use the integrating factor formula:

$$IF = \exp\left(\int \frac{3}{2t}\mathrm{d}t\right) = \exp\left(\frac{3}{2}\log_e t\right) = t^{3/2}$$

Multiply both sides of the equation by the integrating factor:

$$t^{3/2}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3}{2}t^{1/2}x = 6t^2e^{t^3}$$

We see that the DE can now be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(IF.x(t)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(t^{3/2}x(t)\right) = 6t^2e^{t^3}$$

Integrate both sides of the equation:

$$t^{3/2}x(t) = \int 6t^2e^{t^3}$$

Substitute $u = t^3$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = 3t^2$:

$$\begin{aligned} t^{3/2}x(t) &= \int 2e^u\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t \\ &= 2\int e^u\mathrm{d}u = 2e^u + C \\ &= 2e^{t^3} + C \end{aligned}$$

Hence

$$x(t) = 2t^{-3/2}e^{t^3} + Ct^{-3/2}$$

We need to find C. Set $t = 1$ and $x(1) = 2e + 2$:

$$x(1) = 2e + 2 = 2e + C$$

Hence $C = 2$ and our solution is

$$x(t) = 2t^{-3/2}e^{t^3} + 2t^{-3/2}.$$

9. A population of p million after t years is modelled by the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 9p - p^2 - 14$$

   (a) Find the equilibrium solutions and determine whether or not they are stable. Make sure that you give a clear argument for your answer.

   (b) Assume that the initial population is $p = 3.141$. Find the population after t years have elapsed. State what happens as t becomes very large. Show all intermediate steps in your solution of the differential equation. State any constants to 3 decimal places.

   (a) The equilibrium solutions are those that do not change with t. Hence they are equivalent to finding

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 9p - p^2 - 14 = -(p-7)(p-2) = 0$$

   Hence the equilibrium populations are $p = 7$ and $p = 2$.
   From the DE we see that when $p < 2$, $\frac{\mathrm{d}p}{\mathrm{d}t} < 0$. When $2 < p < 7$, $\frac{\mathrm{d}p}{\mathrm{d}t} > 0$. And when $p > 7$, $\frac{\mathrm{d}p}{\mathrm{d}t} < 0$.
   Thus when $p < 2$ the DE implies that p is decreasing in time and so will become smaller and so move away from 2. If p is between 2 and 7 then it will increase towards 7. Finally, when $p > 7$ the DE implies that p is decreasing with time and so p will move towards 7.
   Therefore $p = 2$ is an unstable equilibrium and $p = 7$ is a stable equilibrium.

   (b) We have to solve the DE. It is separable:

$$\frac{1}{p^2 - 9p + 14}\frac{\mathrm{d}p}{\mathrm{d}t} = -1$$

   Integrate both sides. The RHS is simply $-t$, while the LHS becomes

$$\int \frac{1}{p^2 - 9p + 14}\frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}t = \int \frac{\mathrm{d}p}{(p-2)(p-7)}$$

   Do a partial fraction decomposition:

$$\begin{aligned}\frac{1}{(p-2)(p-7)} &= \frac{A}{p-2} + \frac{B}{p-7} \\ &= \frac{A(p-7) + B(p-2)}{(p-2)(p-7)}\end{aligned}$$

   When $p = 7$ the numerators give $1 = 5B$ and when $p = 2$ the numerators give $-5a = 1$. Hence we have

$$\frac{1}{(p-2)(p-7)} = \frac{1}{5(p-7)} - \frac{1}{5(p-2)}$$

Hence the LHS is

$$\begin{aligned}\int \frac{\mathrm{d}p}{(p-2)(p-7)} &= \frac{1}{5}\int \frac{\mathrm{d}p}{p-7} - \frac{1}{5}\int \frac{\mathrm{d}p}{p-2} \\ &= \frac{1}{5}\log_e|p-7| + \frac{1}{5}\log_e|p-2| \\ &= \frac{1}{5}\log_e\left|\frac{p-7}{p-2}\right|\end{aligned}$$

So putting both sides together we have:

$$\frac{1}{5}\log_e\left|\frac{p-7}{p-2}\right| = -t + C$$

Isolate p:

$$\begin{aligned}\log_e\left|\frac{p-7}{p-2}\right| &= -5t + 5C \\ \frac{p-7}{p-2} &= De^{-5t} \qquad\qquad \text{where } D = \pm e^{5C} \\ p - 7 &= Dpe^{-5t} - 2De^{-5t} \\ p(1 - De^{-5t}) &= 7 - 2De^{-5t} \\ p(t) &= \frac{7 - 2De^{-5t}}{1 - De^{-5t}}\end{aligned}$$

where D is some constant.
The initial population is $p(0) = 3.141$. We need to find D:

$$\begin{aligned}p(0) = 3.141 &= \frac{7 - 2D}{1 - D} \\ 3.141(1 - D) &= 7 - 2D \\ D(2 - 3.141) &= 7 - 3.141\end{aligned}$$

Hence $D \approx -3.382$ and so

$$p(t) = \frac{7 + 6.764e^{-5t}}{1 + 3.382e^{-5t}}.$$

When $t \to \infty$, $e^{-5t} \to 0$ and we are left with

$$p(t) \to \frac{7}{1} = 7.$$

So the equilibrium population is 7. We can check this by comparing against our answer to part (a).

10. Five grams of a chemical Q are formed by combining 2 grams of a chemical M and 3 grams of a chemical L. Initially there are 80 grams of M, 300 grams of L and no chemical Q. The rate of formation of Q is proportional to the product of the unconverted amounts of M and L. It is observed that exactly 20 grams of Q have been formed after 5 minutes.

    (a) If $x(t)$ is the amount of chemical Q in grams at time t minutes, write down an equation for $\frac{\mathrm{d}x}{\mathrm{d}t}$ in terms of x and t.

    (b) Find $x(t)$ and hence deduce how much chemical Q is left after 20 minutes.

    (c) How much chemical M is left after a very long time?

    State decimal numbers to 2 decimal places.

    (a) We are told that $\frac{\mathrm{d}x}{\mathrm{d}t} = k(\text{amount of M}) \times (\text{amount of L})$. The amount of M is $80 - \frac{2}{5}x$ and the amount of L is $300 - \frac{3}{5}x$. Hence the DE is

    $$\frac{\mathrm{d}x}{\mathrm{d}t} = k(80 - \frac{2x}{5})(300 - \frac{3x}{5}) = \frac{6k}{25}(200 - x)(500 - x)$$

    where k is some constant yet to be determined.

    (b) We need to solve the DE. It is separable:

    $$\frac{1}{(200 - x)(500 - x)} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6k}{25}$$

    Integrate both sides.

    $$\int \frac{1}{(200 - x)(500 - x)} \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = \int \frac{\mathrm{d}x}{(200 - x)(500 - x)} = \int \frac{6k}{25} \mathrm{d}t = \frac{6kt}{25} + C$$

    Where C is some constant.

    The RHS needs to be written in partial fraction form:

    $$\begin{aligned} \frac{1}{(200 - x)(500 - x)} &= \frac{A}{200 - x} + \frac{B}{500 - x} \\ &= \frac{A(500 - x) + B(200 - x)}{(200 - x)(500 - x)} \end{aligned}$$

    The numerators give $1 = A(500 - x) + B(200 - x)$. Setting $x = 500$ gives $B = -1/300$. Setting $x = 200$ gives $A = 1/300$. Hence

    $$\frac{1}{(200 - x)(500 - x)} = \frac{1}{300} \left(\frac{1}{200 - x} - \frac{1}{500 - x} \right)$$

    We can now do the integral:

    $$\begin{aligned} \int \frac{\mathrm{d}x}{(200 - x)(500 - x)} &= \frac{1}{300} \int \left(\frac{1}{200 - x} - \frac{1}{500 - x} \right) \mathrm{d}x \\ &= \frac{1}{300} (-\log|200 - x| + \log|500 - x|) \\ &= \frac{1}{300} \log \left| \frac{500 - x}{200 - x} \right| \end{aligned}$$

Put it all together:

$$\begin{aligned}\frac{1}{300}\log\left|\frac{500-x}{200-x}\right| &= \frac{6kt}{25}+C\\ \log\left|\frac{500-x}{200-x}\right| &= 72kt+300C\\ \frac{500-x}{200-x} &= \pm\exp(72kt+300C)=D\exp(72kt)\end{aligned}$$

where $D=\pm e^{300C}$

We have to determine 2 constants, k and D. We know that at $t=0$ there is no Q, so $x(0)=0$. Hence

$$\frac{500}{200}=D$$

Therefore we can write

$$\frac{500-x}{200-x}=\frac{5}{2}\exp(72kt)$$

The other piece of information is that at $t=5$ there are 20 grams of Q, ie $x(5)=20$. Substitute this into the equation:

$$\begin{aligned}\frac{500-20}{200-20}=\frac{8}{3} &= \frac{5}{2}\exp(360k)\\ \frac{16}{15} &= \exp(360k)\end{aligned}$$

And so $k=\frac{1}{360}\log(16/15)=0.00018$ or $72k=\frac{1}{5}\log(16/15)\approx 0.013$.
So the solution is

$$\begin{aligned}\frac{500-x}{200-x} &= \frac{5}{2}\exp(72kt)\\ 1000-2x &= 1000e^{72kt}-5xe^{72kt}\\ x(5e^{72kt}-2) &= 1000(e^{72kt}-1)\\ x &= \frac{1000e^{72kt}-1000}{5e^{72kt}-2}\end{aligned}$$

where $72k\approx 0.013$. (this can be stated exactly, but don't need it).
After 20 minutes:

$$\begin{aligned}x(20) &= \frac{1000e^{72k\times 20}-1000}{2e^{72k\times 20}-5}\\ &= \frac{1000e^{0.26}-1000}{5e^{0.26}-2}\\ &= \frac{1000\times 1.297-1000}{5\times 1.297-2}\\ &= \frac{297}{4.485}=66.22\end{aligned}$$

there are 66.22 grams of Q. (rigorous answer is 65.85...— will accept some rounding error).

(c) After a long time $e^{72kt} \to \infty$ since $k > 0$. So we have to do some manipulation to find the limit:

$$\begin{aligned} x &= \frac{1000e^{72kt} - 1000}{5e^{72kt} - 2} \times \frac{e^{-72kt}}{e^{-72kt}} \\ &= \frac{1000 - 1000e^{-72kt}}{5 - 2e^{-72kt}} \end{aligned}$$

Now as $e^{-72kt} \to \infty$ since $-72k < 0$ and we are left with

$$x(t) \to \frac{1000}{5} = 200.$$

Hence there are 200g of Q left after a long time. To make this amount Q we need $200 * 2/5 = 80g$ of M. ie we use up all the M and so there is $0g$ left.

We can check that we haven't run out of L — to make $200g$ of Q requires $80g$ of M and $120g$ of L — this leaves $180g$ of L unused.