Topic 11 — Single population models

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Topic outline

Resources

Some books to look at:

- BZB, Chapter 14
- *Calculus* by Hughes & Hallett, Chapter 10.
- *Differential equations in Mathematical Biology* by D.S. Jones and B.D. Sleeman, Chapters 1, 2, 3, 4, & 8.

Maybe also look at

- *Mathematical ideas in biology* by J.M. Smith.
- Books on DEs in general — call numbers 515.35.

What sort of DEs

- Instead of x and y, we will use t and x
- t will be time measured in some units
  — independent variable
- x will be the population in some units
  — dependent variable
- Our general first order DE is $\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$
- We will consider *autonomous* first order DEs
- These are DEs of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x)$$

ie the RHS is independent of t. The change in the population depends only on the size of the population.

Single population models

- Consider the population x of a "colony" at time t.
- The time t may be in seconds, days, years, ...
- The pop. x may be in units, hundreds, millions, ...
- Choose the scale to make the DE easier
  — after t hours there are x billion bacteria in your gut.
- The variable x should be an integer — can't have 0.43287 of a person. However we treat as continuous.

The models

We will look at 4 simple population models.

1. Doomsday model:
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x$$
Rate of increase of population is proportional to its size. It grows faster and faster.
2. Doomsday model with harvesting:
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \gamma$$
Similar to the above, but the population is harvested at a constant rate.

The models

3. Logistic model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x^2$$

   Similar to Doomsday, but the growth is limited by crowding. As the population approaches some maximum, then the rate approaches zero.

4. Logisitic model with harvesting:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x^2 - \gamma$$

   Similar to the Logisitic model, but the population is harvested at a constant rate.

For all four models:

- the parameters $\alpha, \beta, \gamma > 0$, and
- the initial population $x(0) = x_0 > 0$.

Doomsday model

- Invented by Malthus in 19^{th} C.
- The IVP is $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = & \alpha x \\ x(0) & = & x_0 > 0 \end{cases}$
- x_0 is the initial population
- α is the "specific growth rate"
  — rate of producing offspring.
- $\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha$ — constant "relative" growth rate.

Solve it

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \alpha x \\
\int \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= \int \alpha \mathrm{d}t \\
\int \frac{1}{x}\mathrm{d}x &= \int \alpha \mathrm{d}t \\
\log|x| &= \alpha t + c \\
|x| &= e^c e^{\alpha t} \\
x &= \pm e^c e^{\alpha t}
\end{aligned}$$

- Put $A = \pm e^c$ and

$$x = Ae^{\alpha t}$$

where $A \in \mathbb{R}$ — check the $A = 0$ case.

General and particular

1. The general solution is:

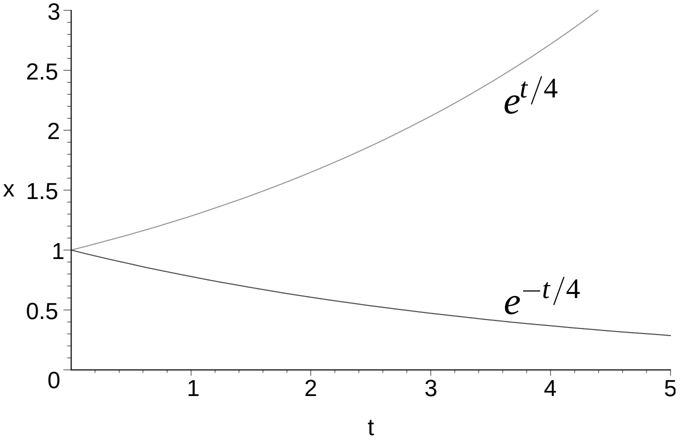
$$x = Ae^{\alpha t} \qquad A \in \mathbb{R}$$

2. Since $x(0) = x_0 = Ae^0 = A$, the particular solution is

$$x = x_0 e^{\alpha t}$$

- What does this tell us?
  - If $x_0 = 0$ then the population will always be zero.
  - If $\alpha > 0$ then population will always $\to \infty$.
  - If $\alpha < 0$ then population will always $\to 0$.

A picture



- If $\alpha > 0$ then population will grow — hence "Doomsday".
- If $\alpha < 0$ then population will decay

Doomsday Model + Harvesting

- Same as Doomsday, but some number of the population dies or is "harvested".

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \gamma$$

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$$\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{rate of increase in population}} = \underbrace{\alpha}_{\text{specific growth rate}} x - \underbrace{\gamma}_{\text{harvest rate}}$$

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$$\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{rate of increase in population}} = \underbrace{\alpha}_{\text{specific growth rate}} x - \underbrace{\gamma}_{\text{harvest rate}}$$

- Three relevant examples:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2 \qquad x(0) = \frac{1}{2}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2 \qquad x(0) = \frac{2}{3}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2 \qquad x(0) = 1$$

General solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \gamma$$

- Is separable and linear.

$$\begin{aligned}
\int \frac{1}{\alpha x - \gamma}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= \int 1\mathrm{d}t \\
\int \frac{1}{\alpha x - \gamma}\mathrm{d}x &= t + c \\
\frac{1}{\alpha}\log|\alpha x - \gamma| &= t + c \\
|\alpha x - \gamma| &= e^{\alpha t}e^{\alpha c} \\
\alpha x &= \gamma \pm e^{\alpha t}e^{\alpha c} \\
x &= \frac{\gamma}{\alpha} + Ae^{\alpha t}
\end{aligned}$$

where $A \in \mathbb{R}$ — check $A = 0$ case.

Particular solutions

The DE $\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2$ has general solution

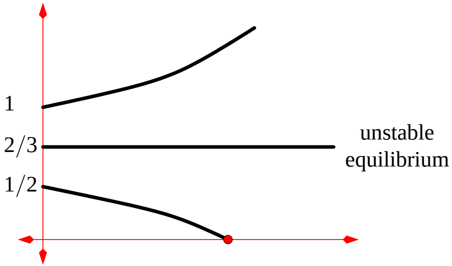
$$x = \frac{2}{3} + Ae^{3t}$$

1. IC $x(0) = \frac{1}{2}$ implies $\frac{1}{2} = \frac{2}{3} + A$ so $A = -\frac{1}{6}$
$$x = \frac{2}{3} - \frac{1}{6}e^{3t}$$
2. IC $x(0) = \frac{2}{3}$ implies $\frac{2}{3} = \frac{2}{3} + A$ so $A = 0$
$$x = \frac{2}{3}$$
3. IC $x(0) = 1$ implies $1 = \frac{2}{3} + A$ so $A = \frac{1}{3}$
$$x = \frac{2}{3} + \frac{1}{3}e^{3t}$$

The examples

- These three solutions of the same DE but with different IC give quite different behaviour.
- Look again at the DE:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &> 0 \qquad \text{if } x > \frac{2}{3} \qquad \text{growth} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \qquad \text{if } x = \frac{2}{3} \qquad \text{stable} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &< 0 \qquad \text{if } x < \frac{2}{3} \qquad \text{decay} \end{aligned}$$



A bit more detail

1. In example 1 there is a time $t = T$ when the population dies out.

$$0 = \frac{2}{3} - \frac{1}{6}e^{3T}$$

$$e^{3T} = 4 \qquad \text{or } T = \frac{1}{3}\log_e 4$$

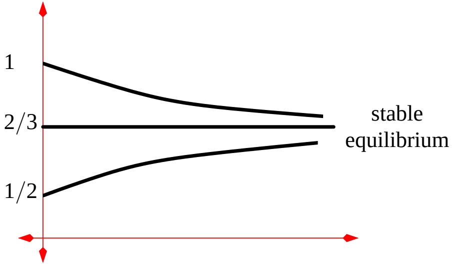
2. In example 2, $x(t) = \frac{2}{3}$ — a constant.
   In this case $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ for all t
   Such solutions are called "equilibrium solutions".
3. In example 3, $x(t) = \frac{2}{3} + \frac{1}{3}e^{3t}$. As $t \to \infty$ $x(t) \to \infty$.
   So we have essentially Doomsday growth.
   Not enough harvesting.

An aside...

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x + 2$$

- Chemical reaction or mixing model ($\alpha > 0$)

$$\begin{array}{llll} \frac{\mathrm{d}x}{\mathrm{d}t} < 0 & \text{if } x > \frac{2}{3} & \text{decay} \\ \frac{\mathrm{d}x}{\mathrm{d}t} = 0 & \text{if } x = \frac{2}{3} & \text{stable} \\ \frac{\mathrm{d}x}{\mathrm{d}t} > 0 & \text{if } x < \frac{2}{3} & \text{growth} \end{array}$$



Equilibria

- An "equilibrium solution" — solution for which $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. The solution does not change with time.
- We classify equilibria according to what happens to "nearby" solutions.
- A "stable equilibrium" — solutions that start from IC close to the equilibrium move closer as $t \to \infty$.
- An "unstable equilibrium" — solutions that start from IC close to the equilibrium move further away as $t \to \infty$.

How to classify

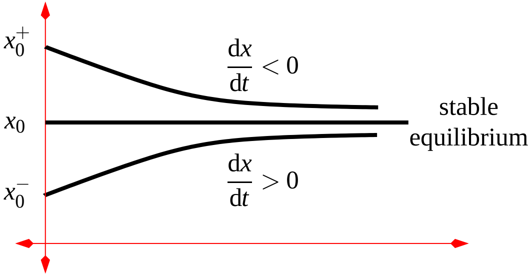
- Consider the IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x) \qquad \text{and } x(0) = x_0$$

- Say the solution is $x(t) = x_0$, so $\frac{\mathrm{d}x}{\mathrm{d}t} = G(x_0) = 0$ for all t — an equilibrium solution.
- "Stable" means nearby curves move closer.
- "Unstable" means nearby curves move further away.
- Let us consider two nearby IC and see how they must move if the equilibrium solution is stable or unstable.

Stable equilibrium

- Let us say that $x(t) = x_0$ is a stable equilibrium.
- Consider two other IC x_0^- and x_0^+ such that $x_0^- < x_0 < x_0^+$.

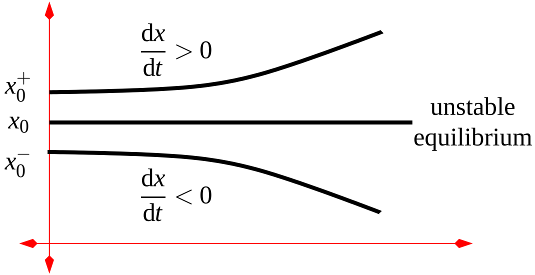


- The curve starting at x_0^+ must decrease towards x_0.
- The curve starting at x_0^- must increase towards x_0.

$$\left.\begin{array}{lcll} \frac{dx}{dt} & < & 0 & \text{if } x > x_0 \\ \frac{dx}{dt} & > & 0 & \text{if } x < x_0 \end{array}\right\} \text{stable equilibrium}$$

Unstable equilibrium

- Let us say that $x(t) = x_0$ is an unstable equilibrium.
- Consider two other IC x_0^- and x_0^+ such that $x_0^- < x_0 < x_0^+$.



- The curve starting at x_0^+ must increase away from x_0.
- The curve starting at x_0^- must decrease away from x_0.

$$\left.\begin{array}{lll} \frac{dx}{dt} > 0 & \text{if } x > x_0 \\ \frac{dx}{dt} < 0 & \text{if } x < x_0 \end{array}\right\} \text{unstable equilibrium}$$

How to find and classify equilibria

- We are given a DE of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x)$$

- Find equilibrium points by solving $G(x) = 0$.
- If asked about their stability then look at sign of $G(x)$
  - for x close to x_0 but bigger, and
  - for x close to x_0 but smaller.
- We will either get

$$\left.\begin{array}{lll} \frac{\mathrm{d}x}{\mathrm{d}t} < 0 & & \text{if } x > x_0 \\ \frac{\mathrm{d}x}{\mathrm{d}t} > 0 & & \text{if } x < x_0 \end{array}\right\} \text{stable equilibrium}$$

or otherwise

$$\left.\begin{array}{lll} \frac{\mathrm{d}x}{\mathrm{d}t} > 0 & & \text{if } x > x_0 \\ \frac{\mathrm{d}x}{\mathrm{d}t} < 0 & & \text{if } x < x_0 \end{array}\right\} \text{unstable equilibrium}$$

Logistic Model

- Takes into crowding into account.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x^2 \qquad \alpha, \beta > 0$$

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$$\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{rate of increase in population}} = \underbrace{\alpha}_{\text{specific growth rate}} x - \underbrace{\beta}_{\text{crowding parameter}} x^2$$

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$$\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}}_{\substack{\text{rate of}\\ \text{increase in}\\ \text{population}}} = \underbrace{\alpha}_{\substack{\text{specific}\\ \text{growth}\\ \text{rate}}} x - \underbrace{\beta}_{\substack{\text{crowding}\\ \text{parameter}}} x^2$$

- An example:
  - Find the equilibrium solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 4x^2$$

    and describe their stability.
  - Then find $x(t)$ if $x(0) = 1$ and if $x(0) = 2$.

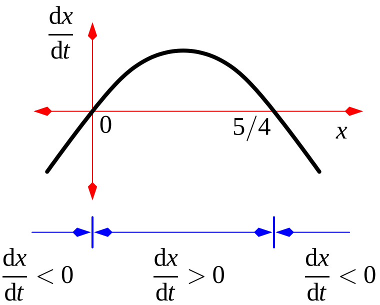
The example

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 4x^2$$

- To find the equilibrium solutions we solve $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$.
- So we solve $5x - 4x^2 = x(5 - 4x) = 0$.
- Equilibrium solutions are $x = 0, \frac{5}{4}$.
- Need to work out their stability
- Perhaps easiest to plot $\frac{\mathrm{d}x}{\mathrm{d}t}$ against x.

Stability

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 4x^2$$



- If x is close to 0 it moves away — unstable.
- If x is close to $5/4$ it moves closer — stable.

General solution

- It is separable:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 5x - 4x^2 \\
\int \frac{1}{x(5-4x)} \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t &= \int 1 \mathrm{d}t + c \\
\int \frac{1}{x(5-4x)} \mathrm{d}x &= t + c \\
\int \left(\frac{1}{5x} + \frac{4}{5(5-4x)} \right) \mathrm{d}x &= t + c \\
\frac{1}{5} \log|x| - \frac{1}{5} \log|5-4x| &= t + c \\
\log\left| \frac{x}{5-4x} \right| &= 5t + 5c
\end{aligned}
$$

General solution

- Keep going...

$$
\begin{aligned}
\log\left|\frac{x}{5-4x}\right| &= 5t+5c \\
\left|\frac{x}{5-4x}\right| &= e^{5c}e^{5t} \\
\frac{x}{5-4x} &= Ae^{5t} \qquad\qquad A=\pm e^{5c}
\end{aligned}
$$

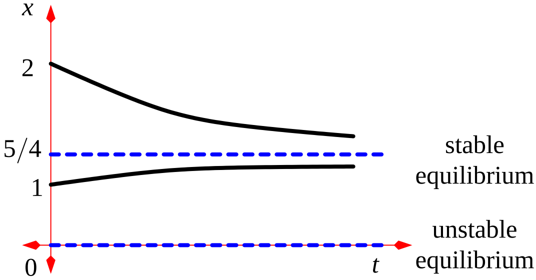
Solving for $x(t)$ gives:

$$
x(t)=\frac{5Ae^{5t}}{4Ae^{5t}+1}=\frac{5A}{4A+e^{-5t}}
$$

Particular solutions

$$\frac{x}{5-4x} = Ae^{5t} \qquad \text{or} \qquad x(t) = \frac{5A}{4A + e^{-5t}}$$

- If $x(0) = 1$ then $A = 1 \longrightarrow x(t) = \dfrac{5}{4 + e^{-5t}}$.
- If $x(0) = 2$ then $A = -\frac{2}{3} \longrightarrow x(t) = \dfrac{10}{8 - 3e^{-5t}}$.
- Both solutions $\to 5/4$ as $t \to \infty$.



Phase line

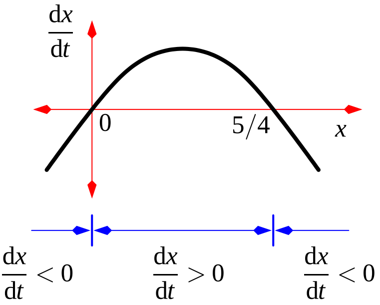
- Consider the general autonomous DE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x)$$

- The RHS is independent of t.
- So the slopefield is independent of t
- No need to draw 2-d picture of slopefield to get an idea of the shape of solutions.
- Instead draw a "phase line".
- 1-d picture showing the equilibrium points and arrows to show sign of $\frac{\mathrm{d}x}{\mathrm{d}t}$.

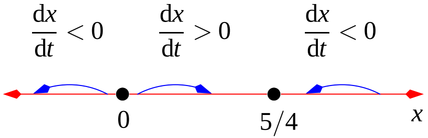
Phase line

- In the previous example $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 5x - 4x^2$.
- Take the plot of $\frac{\mathrm{d}x}{\mathrm{d}t}$ against x



Phase line

- In the previous example $\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 4x^2$.
- Take the plot of $\frac{\mathrm{d}x}{\mathrm{d}t}$ against x
- and condense it down to a “phase line”



- We can look at the phase line and see how a solution will move.
- Gives the shape of a solution.

Logistic model with harvesting

- Takes into crowding into account.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x^2 - \gamma \qquad \alpha, \beta, \gamma > 0$$

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$$\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{rate of increase in population}} = \underbrace{\alpha}_{\text{specific growth rate}} x - \underbrace{\beta}_{\text{crowding parameter}} x^2 - \underbrace{\gamma}_{\text{harvest rate}}$$

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- An aside:
  - If $\alpha^2 - 4\beta\gamma > 0$ then there are 2 equilibrium solutions.
  - If $\alpha^2 - 4\beta\gamma < 0$ then there no equilibrium solutions.

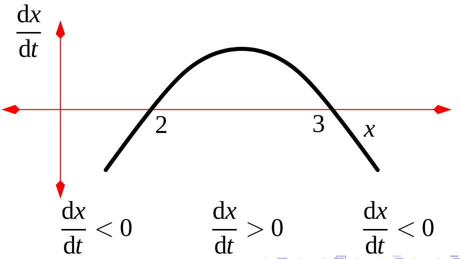
An example

- An example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - x^2 - 6 \qquad x(0) = x_0$$

  - Population of x thousand birds in a park.
  - Culled at a rate of 6 thousand per year.
- Find and classify the equilibrium points.

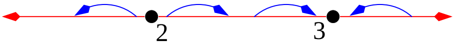
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x-2)(x-3)$$



Equilibrium points

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x-2)(x-3)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} < 0 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} > 0 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} < 0$$



- So $x = 2$ is an unstable equilibrium point,
- And $x = 3$ is an stable equilibrium point.

General solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x-2)(x-3)$$

- Equation is separable

$$\begin{aligned}
\int \frac{1}{(x-2)(x-3)} \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= -\int 1\mathrm{d}t + c \\
\int \frac{1}{(x-2)(x-3)}\mathrm{d}x &= c - t \\
\int \left(\frac{-1}{x-2} + \frac{1}{x-3} \right) \mathrm{d}x &= c - t \\
-\log|x-2| + \log|x-3| &= c - t \\
\log\left|\frac{x-3}{x-2}\right| &= c - t \\
\frac{x-3}{x-2} &= Ae^{-t} \qquad A = \pm e^{-c}
\end{aligned}$$

General solutions

- Solve for $x(t)$:

$$\begin{aligned}
\frac{x-3}{x-2} &= Ae^{-t} \\
x-3 &= Ae^{-t}(x-2) \\
x\left(1-Ae^{-t}\right) &= 3-2Ae^{-t} \\
x &= \frac{3-2Ae^{-t}}{1-Ae^{-t}}
\end{aligned}$$

Particular solution

- To find A use $\frac{x-3}{x-2} = Ae^{-t}$. Substitute in $x(0) = x_0$:

$$\frac{x_0 - 3}{x_0 - 2} = Ae^0 = A$$

- So $x(t)$ is:

$$x = \frac{3 - 2\left(\frac{x_0-3}{x_0-2}\right)e^{-t}}{1 - \left(\frac{x_0-3}{x_0-2}\right)e^{-t}}$$

Can the population die out?

- Can the population die out? Solve $x(T) = 0$.

$$\begin{aligned} 3 &= 2\left(\frac{x_0 - 3}{x_0 - 2}\right) e^{-T} \\ \frac{3}{2}e^{T} &= \left(\frac{x_0 - 3}{x_0 - 2}\right) \\ T &= \log \frac{2}{3}\left(\frac{x_0 - 3}{x_0 - 2}\right) \end{aligned}$$

- Clearly depends on x_0.
- Cannot have $x_0 = 2$ — this is the unstable equilibrium.
- Also must have $T \geq 0$ so argument of log > 1.

Death of the population

- We have 3 cases to consider:

1. If $x_0 > 3$ then
   - $x_0 - 3 > 0$ and $x_0 - 2 > 0$ so $\frac{2}{3}\left(\frac{x_0-3}{x_0-2}\right) > 0$.
   - But $(x_0 - 3) < (x_0 - 2)$ so $\frac{2}{3}\left(\frac{x_0-3}{x_0-2}\right) < 1$.
   - Negative time — no population death.
2. If $2 < x_0 \leq 3$ then
   - $(x_0 - 3) \leq 0$ and $(x_0 - 2) > 0$ so $\frac{2}{3}\left(\frac{x_0-3}{x_0-2}\right) < 0$.
   - Time is the log of a negative number — no population death.

Death of the population

3. If $x_0 < 2$ then
   - $(x_0 - 3) < 0$ and $(x_0 - 2) < 0$ so $\frac{2}{3}\left(\frac{x_0-3}{x_0-2}\right) > 0$.
   - But is it bigger than 1? $\frac{2x_0-6}{3x_0-6} > 1$?
   - $6 - 2x_0 > 6 - 3x_0$ so all is fine.

- Hence population death can occur if $x_0 < 2$ and it occurs at

$$T = \log \frac{2}{3}\left(\frac{x_0 - 3}{x_0 - 2}\right)$$

Linearising a DE about an equilibrium

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x)$$

- Say that $x = x_e$ satisfies $G(x_e) = 0$.
- So that x_e is an equilibrium solution of the above DE.
- We approximate $G(x)$ near $x = x_e$ by substituting $x = u + x_e$:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}u}{\mathrm{d}t} \qquad \text{so} \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= G(u + x_e) \end{aligned}$$

Linearising a DE about an equilibrium

$$\frac{\mathrm{d}u}{\mathrm{d}t} = G(u + x_e) \qquad u(0) = u_0$$

- We approximate $G(u + x_e)$ by it Taylor expansion about $x = x_e$ — only need the linear term.
- The higher order terms $(u^2, u^3, \dots)$ are very small
- Linear approximation is:

$$P_1(u) = G(x_e) + uG'(x_e) = 0 + bu$$

- So our DE becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} \approx bu$$

Test for stability

$$\frac{\mathrm{d}u}{\mathrm{d}t} \approx bu$$
$$u \approx u_0 e^{bt}$$

- So $x \approx x_e + u_0 e^{bt}$.
- What happens as t becomes large?
- All depends on b:
- If $b < 0$ then $|u_0 e^{bt}|$ becomes small and $x \to x_e$ — a stable equilibrium.
- If $b > 0$ then $|u_0 e^{bt}|$ becomes large and $x \not\to x_e$ — an unstable equilibrium.

An example

- Let us return to the IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - x^2 - 6 \qquad x(0) = x_0$$

- Suppose x_0 is "near" the equilibrium $x = 2$.
- Make the substitution $x = u + 2$ with $|u| \ll 1$ small.
- Then the DE becomes

$$\begin{aligned} \frac{\mathrm{d}(u+2)}{\mathrm{d}t} &= 5(u-2) - (u-2)^2 - 6 \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= 5u - 10 - (u^2 - 4u + 4) - 6 \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= u - u^2 \end{aligned}$$

Continued...

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u - u^2 \qquad |u_0| \ll 1$$

- Since x_0 is "near" x=2, u_0 is "near" $u = 0$
- Since u_0 is small, u_0^2 is much smaller
- So disregard the u^2 term — leave the linear term (Taylor approximation)

$$\frac{\mathrm{d}u}{\mathrm{d}t} \approx u \qquad \text{linearised around } u = 0$$

- The solution is $u = u_0 e^t$
- Hence $|u|$ gets bigger as $t \to \infty$.
- Hence the point $x = 2$ is an unstable equilibrium.

The other equilibrium

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - x^2 - 6 \qquad x(0) = x_0$$

- Suppose x_0 is "near" the equilibrium $x = 3$.
- Make the substitution $x = v + 3$ with $|v| \ll 1$ small.
- Then by similar workings the DE becomes

$$\begin{aligned} \frac{\mathrm{d}(v+3)}{\mathrm{d}t} &= 5(v-3) - (v-3)^2 - 6 = -v - v^2 \\ \frac{\mathrm{d}v}{\mathrm{d}t} &\approx -v \qquad v(0) = v_0 \end{aligned}$$

- This has solution $v = v_0 e^{-t}$.
- This goes to 0 as t gets large — stable equilibrium