Topic 3: Solving Linear Programming problems graphically.

BZB 5.1 and 5.2.

Exploratory Example

Two brands of cough syrup (A and B) can be mixed to make up a dose for a patient. Brand A contains 2 milligrams per millilitre of a drug, and Brand B contains 3 mg per ml of the same drug. A dose must contain no more than 12 mg of the drug.
(A 'word' problem)

Investigate

Suppose a dose consists of

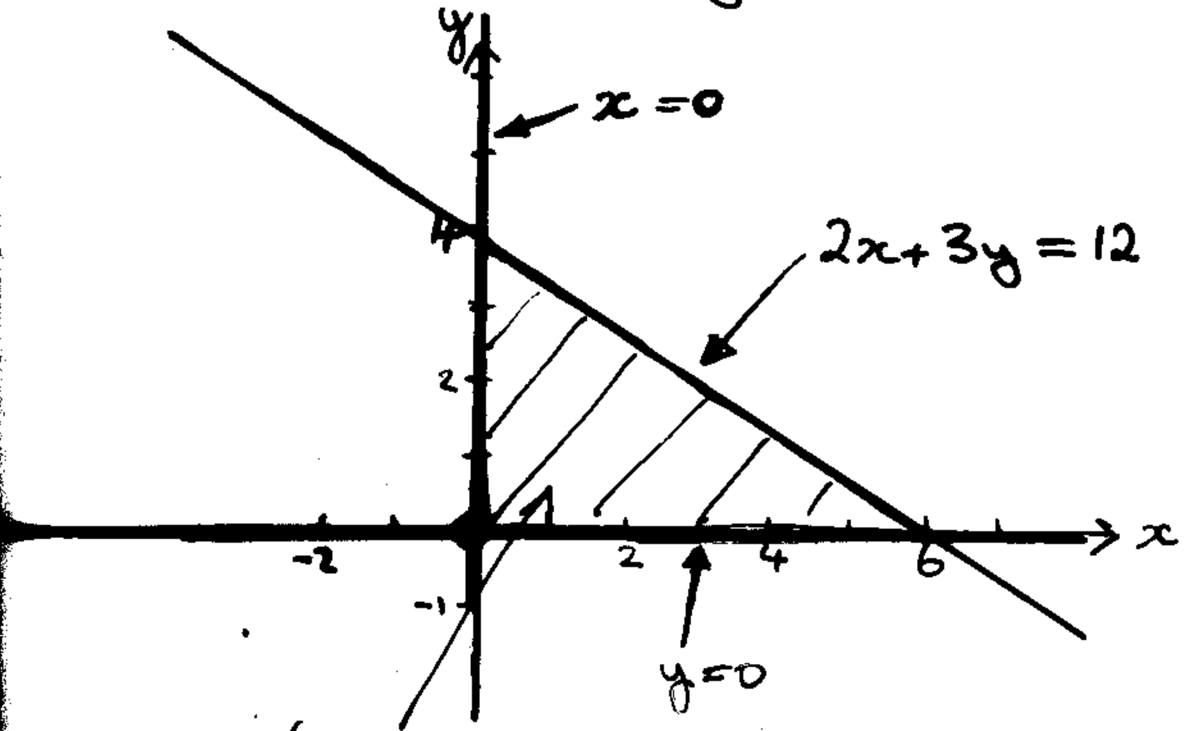
x ml of A and

y ml of B

Then the dose has $2x+3y$ ml of the drug.

So

$$2x + 3y \leqslant 12$$
$$x \geqslant 0$$
$$y \geqslant 0$$



'feasible' region (includes boundaries)

So (x, y) must be in the triangular shaded region called 'the feasible region'.

End Investigation. (Note that by default $x \leq 6$ and $y \leq 4$)

Constraints lead to inequalities which give a 'feasible region'.

An aside: We could write the 'above' set of inequations as

$$2x + 3y + s = 12$$

with restrictions $x \geqslant 0$, $y \geqslant 0$ and $s \geqslant 0$

(An underdetermined system: 1 eqn, 3 variables)

We do not need to state that $x \leq 6$, $y \leq 4$ and $s \leq 12$

$s \leftrightarrow$ something positive, sometimes called 'the slack'.

We will return to the idea of writing an inequation as an equation with an extra variable (plus a simple constraint eg $s \geqslant 0$)

Sketching half-planes

Ex (i) $\quad x + 4y \leq 8$

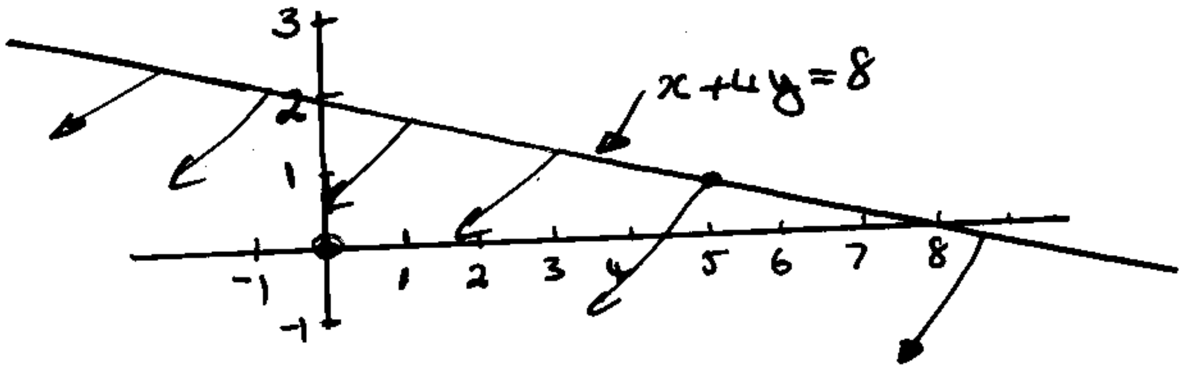
- Draw the line (the equality) $x+4y=8$
- Decide which side is the half plane

or

look at a point not on the line : eg $(0,0)$

rewrite inequality as $x \leq 8-4y$

To the left.



Ex(ii) $x - 2y \geqslant 4$

• Draw the line $x - 2y = 4$

Putting $x=0$ gives $y=-2$

Putting $y=0$ gives $x=4$

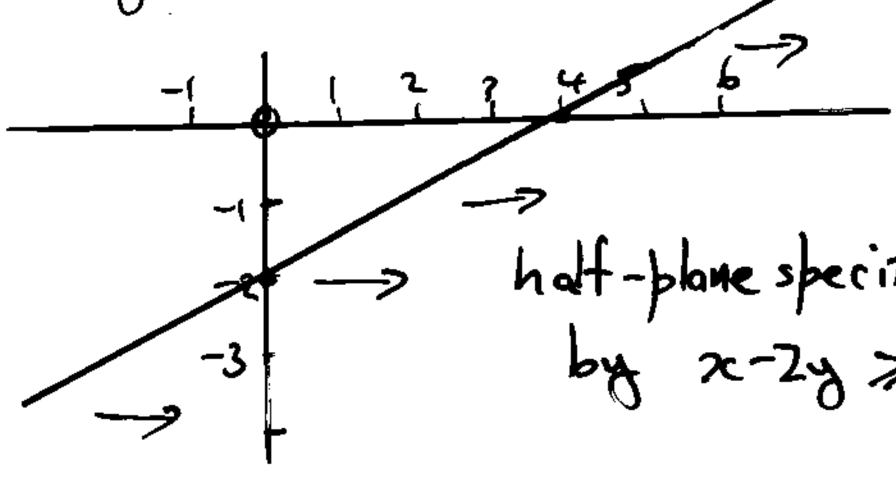
So $(0,-2)$ and $(4,0)$ are two points on the line $x-2y=4$ and we join the dots...

• Which side

method 1: look at $(0,0)$ $0-2\times 0 = 0$

and $0 \ngeqslant 4$ so our half-plane doesn't include $(0,0)$.

method 2: we write $x \geqslant 4-2y$ so x values are to the right ($\geqslant$) of the line



half-plane specified by $x-2y \geqslant 4$.

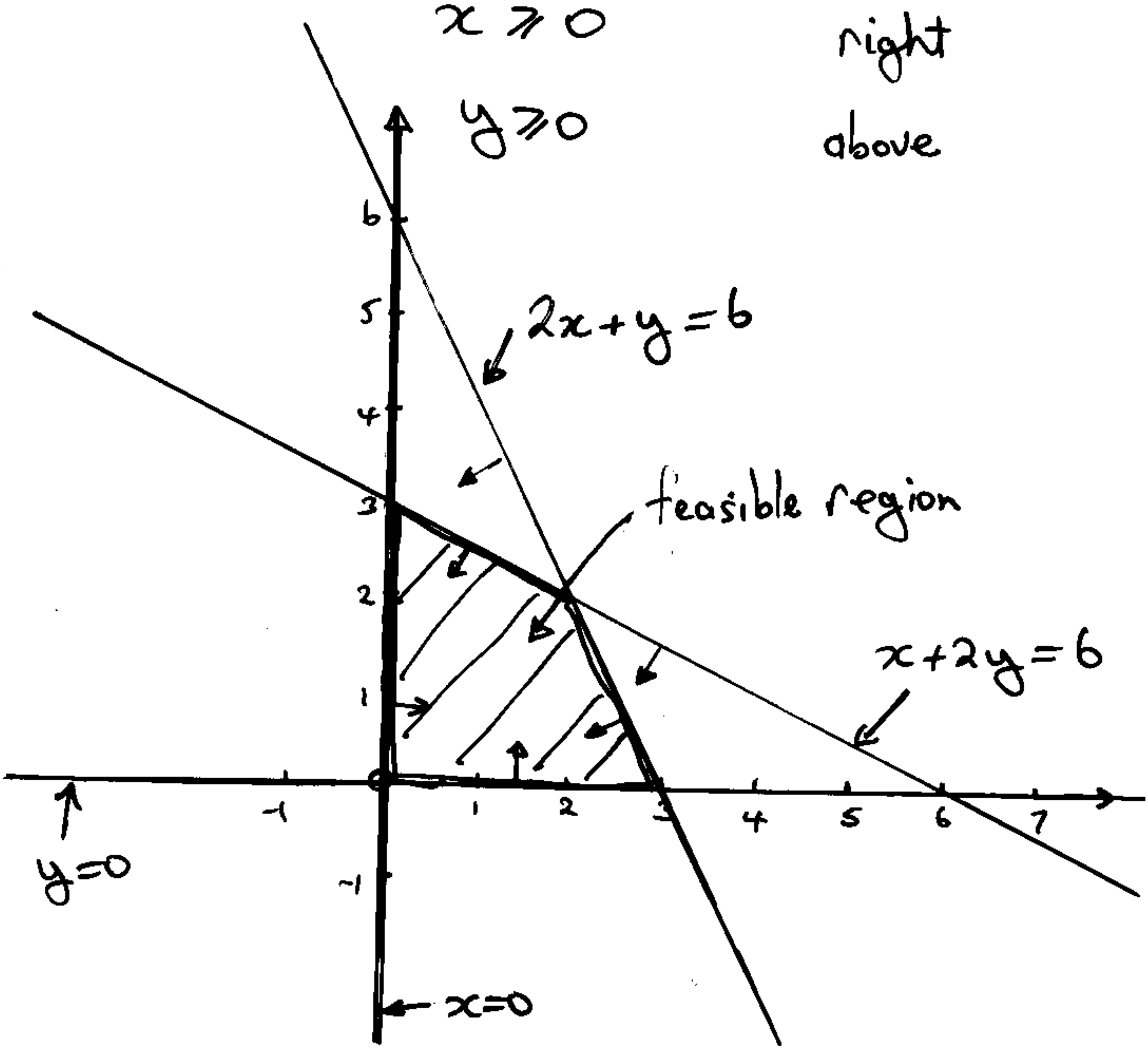
Ex (iii) Simultaneous inequations

$x + 2y \leq 6$ left

$2x + y \leq 6$ left

$x \geq 0$ right

$y \geq 0$ above



So (x, y) can be chosen to be any point within the shaded //// region

A Lineor Programming problem concerns the optimisation (ie finding the maximum or minimum value of) of a linear function (function that is linear in all variables)

subject To a set of linear problem constraints (set of linear inequalities)

Usually all variables Take on only non-negative values.

The variables are called decision variables.

We will deal with 2 standard forms.

First: The standard maximum problem.

620-151 BIOMEDICAL MATHEMATICS

Lecture Supplement

EXAMPLE to introduce linear programs (see similar introductory example in B & Z6 pages 274 – 277 or B & Z7 pages 279 - 282)

This is our first 'Linear Programming' problem

PROBLEM:

A small manufacturer has just one sewing machine and one press/stretch machine on which to make shirts and jackets. Each shirt requires four minutes sewing and one minute pressing/stretching, and each jacket requires two minutes sewing and three minutes pressing/stretching. The sewing machine can only operate for 33 hours 20 minutes per week owing to the restricted availability of the operator, and the press/stretch machine can only operate for 25 hours per week, as it needs frequent servicing. The manufacturer knows that there is enough demand to sell as many shirts and jackets as can be made each week. The profit made is \$5 per shirt and \$6 per jacket. How many shirts and jackets should the manufacturer make each week to maximise profit?

MATHEMATICAL MODEL:

Decision Variables

Suppose that in a given week, x shirts and y jackets are made.

Note that 33 hours 20 mins = 2000 minutes; and 25 hours = 1500 minutes.

So (fill in the inequations):

sew: $4x + 2y \leq 2000$

press: $x + 3y \leq 1500$

$x \geq 0$

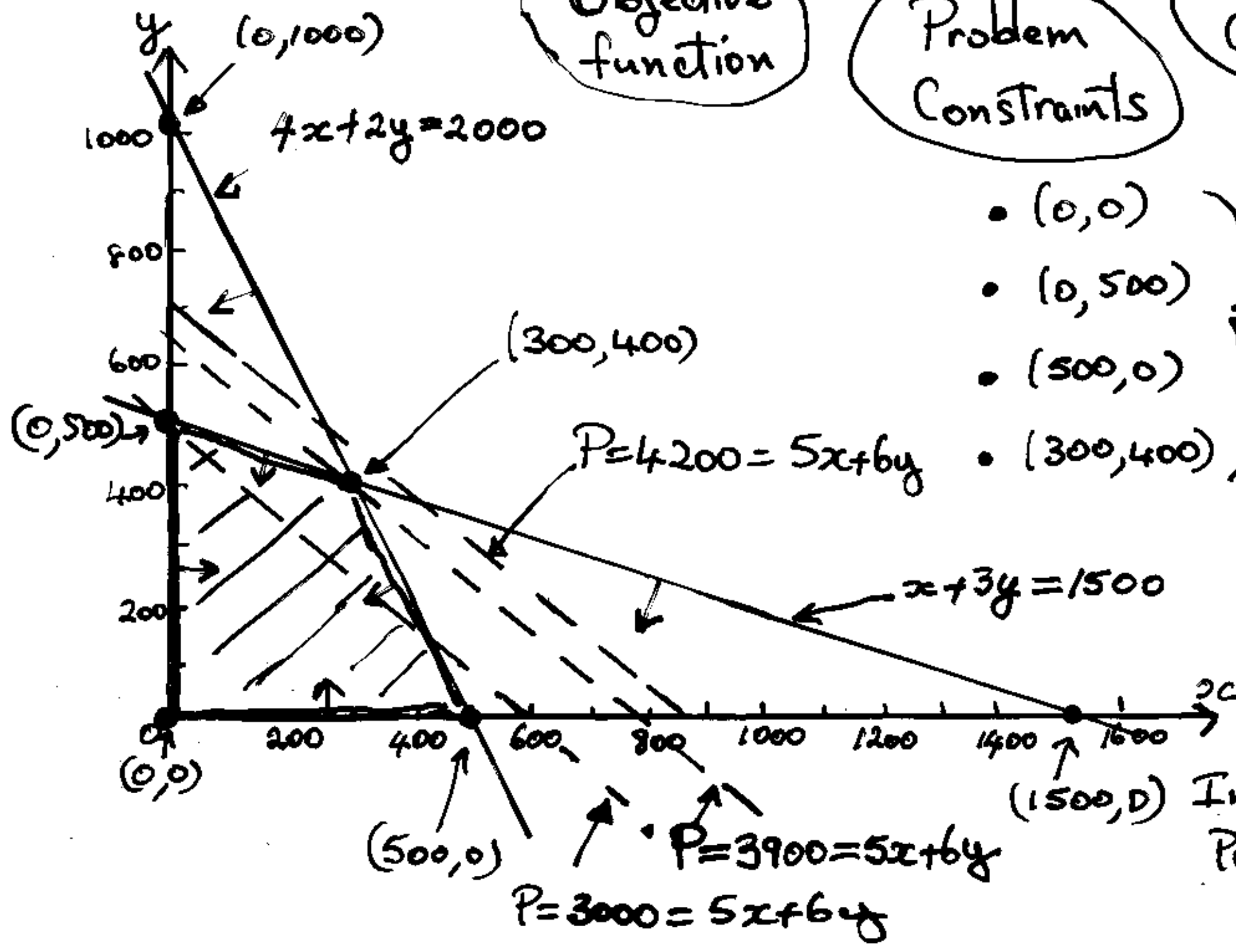
$y \geq 0$

The weekly profit is $P = 5x + 6y$.

Objective function

Problem Constraints

Non-negativity Constraints



- (0,0)
- (0,500)
- (500,0)
- (300,400)

Corner Points of Feasible region

620-151

Introduction to Biomedical Mathematics

Linear Programming Jargon List

(see B & Z6 pages 259 – 336 or B & Z7 pages 265 - 339)

In the linear programming topics, the following terms will be introduced and used on many occasions. While you need to have some acquaintance with the terminology, it is better to know what to do rather than to memorise the formal definitions!

Linear Programming . . . . **Graphics**

(n variable - Simplex) **(2 variable only)**

mathematical model

objective function . . . . function constant line

decision variables (x_1, x_2, etc.)

problem constraints . . . . half planes

non-negativity constraints

standard maximum problem
(maximise P - all problem constraints with $\leq$)

standard minimum problem
(minimise P - all problem constraints with $\geq$)

optimal solution ← what we want: the best solution.

slack variables (s_1, s_2, etc.)

basic variables

non-basic variables

basic solutions . . . . intersection points

feasible solutions . . . . feasible region

basic feasible solutions . . . . corner points of feasible region

simplex tableau

pivot column (negative entry in last row)

pivot row (smallest quotient etc.)

pivot element

pivot operations

initial basic feasible solution

entering variable* (non-basic becoming basic)

exiting variable* (basic becoming non-basic)

In 620-151, we consider *standard maximum* and *standard minimum* problems only, as an application of row operations. There are complications if a problem has a *mixture* of problem constraints (i.e. $\leq$, $\geq$, $=$). The second year subject 620-261 addresses these sorts of mixed constraint problems.

From diagram To solution.

1. (x,y) must be in the shaded region, or on the edge of it. The corner points are (obviously) $(0,0)$, $(500,0)$, $(0,500)$ and (not so obviously) $(300, 400)$

To find the intersection of
$$4x+2y=2000$$
$$x+3y=1500$$

either read off solution from the graph or (better) calculate

$$\left[\begin{array}{cc|c} 4 & 2 & 2000 \\ 1 & 3 & 1500 \end{array}\right] \sim \ldots \text{etc} \sim \left[\begin{array}{cc|c} 1 & 0 & 300 \\ 0 & 1 & 400 \end{array}\right]$$

2. Why (x,y) gives the maximum of P?

Let us sketch $5x+6y=3000$. Part of that straight line lies within the shaded (feasible) region, and we could take any (x,y) on that line segment, always with $P=3000$. Clearly, there are feasible points to the left of the line with $P<3000$, and points to the right for which $P > 3000$.

• Let us sketch $5x + 6y = 4200$. Clearly, no point on this is feasible.

• Let us sketch $5x + 6y = 3900$. One point on this line, $(300, 400)$, is feasible. There are feasible points to the lower left but then $P < 3900$. There ar points to the upper right for which $P > 3900$ but none are feasible.

We conclude that $\max P = 3900$

when $(x, y) = (300, 400)$

* Note that the optimal solution lies on the boundary of the feasible region. This is always the case if a solution exists <u>but</u> a solution doesn't always exist.

* How does one set out a solution:
No need to write an essay but must argue case. See summary next...

* A feasible region is bounded if you can draw a circle around the region. Bounded feasible regions in linear programming problems always have optimal solutions. These always lie on the boundary of the feasible region.

Summary of "graphical method" for solving linear programming (LP) problems.

* Draw a graph with 'problem constraint' lines and so find the feasible region using the inequalities (including non-negativity)

* Draw some objective function constant lines → Discover which part of the boundary may give optimal (maximum) solution: This is related to the 'movement' of objective function value within region.

* Find (by row reduction or otherwise) the corner points near where you suspect the maximum to be. This should lead you to find the optimal objective function straight line.

* Find objective function at corner points

There may be a 'tie' between adjacent corner points : This implies that the whole line segment between those corner points is a segment of optimal solutions → This means we need to specify this segment → parametrically.

Also, the feasible region may be <u>unbounded</u> This implies that there <u>may</u> or <u>may not</u> be any optimal solutions.
Be careful.

Ex 3. Minimise $f = x + 7y$

subject to the constraints

$$2x + 3y \geqslant 7$$
$$5x + 4y \geqslant 14$$
$$x \geqslant 0, \quad y \geqslant 0$$

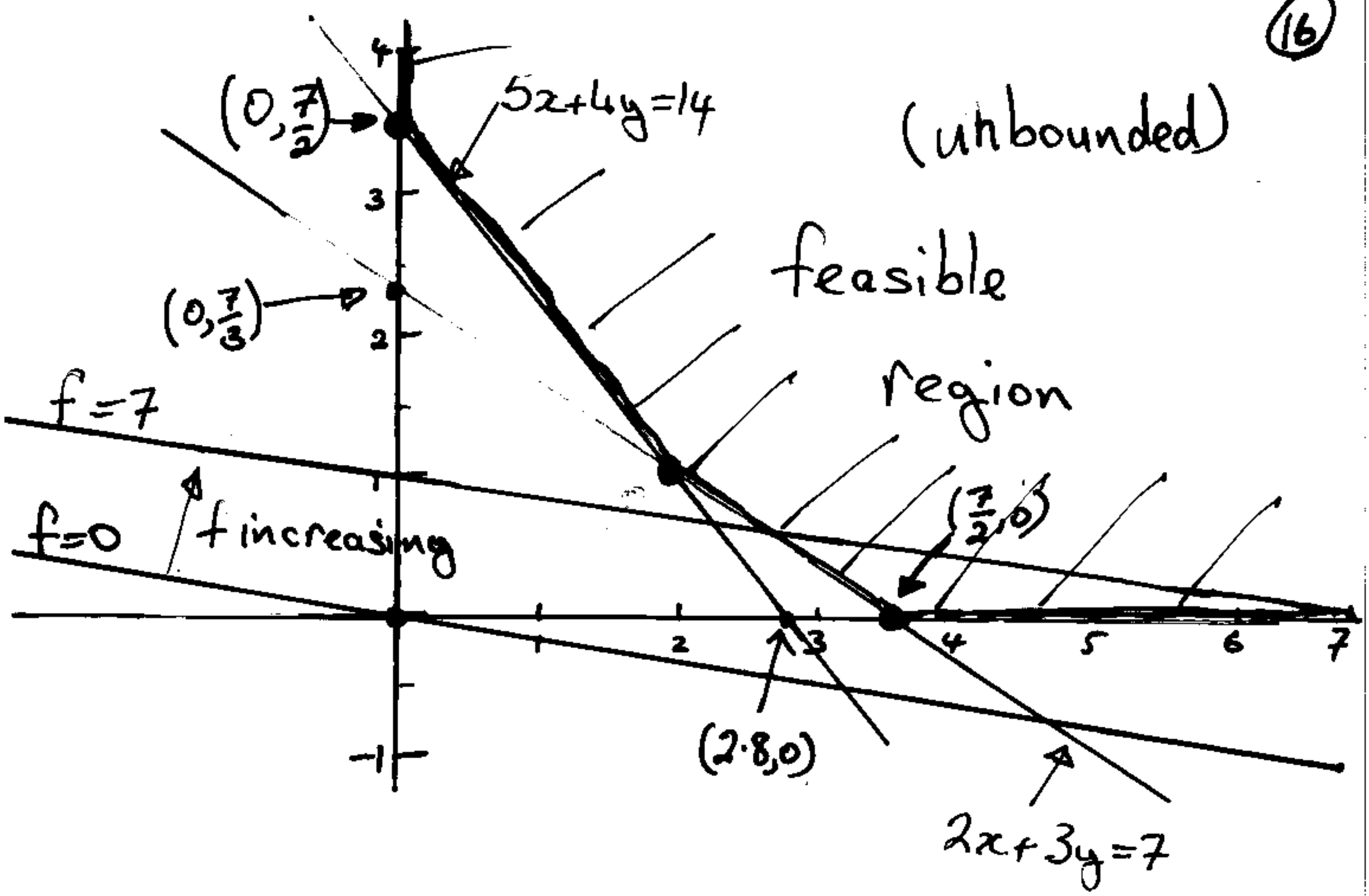
Standard minimum problem

Graphical solution:

Constraint boundaries are
$$x = 0$$
$$y = 0$$
$$2x + 3y = 7$$
$$5x + 4y = 14$$

The objective function straight lines are

$$y = -\frac{1}{7}x + C \quad \text{where } C = \frac{f}{7}$$



Try $f=0$ and $f=7$.

$f=0$ give objective function straight line

$$x+7y=0 \quad \text{ie: } \quad y=-\frac{1}{7}x$$

$f=7$ gives $x+7y=7$ ie $y=-\frac{1}{7}x+1$

We notice how the line moves as we change f and notice that the corner at $(3\frac{1}{2}, 0)$ is The lowest point Touch by any line $f =$ constant $= x + 7y$

"We can argue This since $f = 0$ is completely outside The feasible region while The line $f = 7$ crosses The interior of The feasible region and The only corner point beTween These 2 lines is $(3\frac{1}{2}, 0)$".

Now, when $(x, y) = (3\frac{1}{2}, 0)$ $f = 3\frac{1}{2} + 7 \times 0 = 3\frac{1}{2}$

so The constant objective function sTraight line $3\frac{1}{2} = x + 7y$ ie $y = -\frac{1}{7}x + \frac{1}{2}$

goes Through The corner point $(3\frac{1}{2}, 0)$

Since any other line (other values of f) that goes through (or Touches) The feasible region has a <u>larger</u> value of f we have found our answer, since all lines going Through The feasible region lie To The upper, right of $f = 3\frac{1}{2}$.

Hence, The answer is:

Minimum of f occurs at $(x, y) = (3\frac{1}{2}, 0)$ with $f = 3.5$.

Graphically, we can see, and argue, that we do not need to find other corner points' coordinates (or values of f at them), since they lie to the upper right of the line that runs through $(3\frac{1}{2}, 0)$.

eg: At the corner point $(0, 3\frac{1}{2})$

$$f = 7 \times 3\tfrac{1}{2} = \frac{49}{2} = 24\tfrac{1}{2} > \frac{7}{2}$$

while the other corner point is at the intersection of

$$\left.\begin{aligned} 2x + 3y &= 7 \\ 5x + 4y &= 14 \end{aligned}\right\}$$

which is at $(2, 1)$, and at this point $f = 9 > \frac{7}{2}$

Note that if we were asked about the maximum of f we need to ask about 'unbounded' solutions

Ex4 (Multiple optimal solutions - BZB7 285→287)

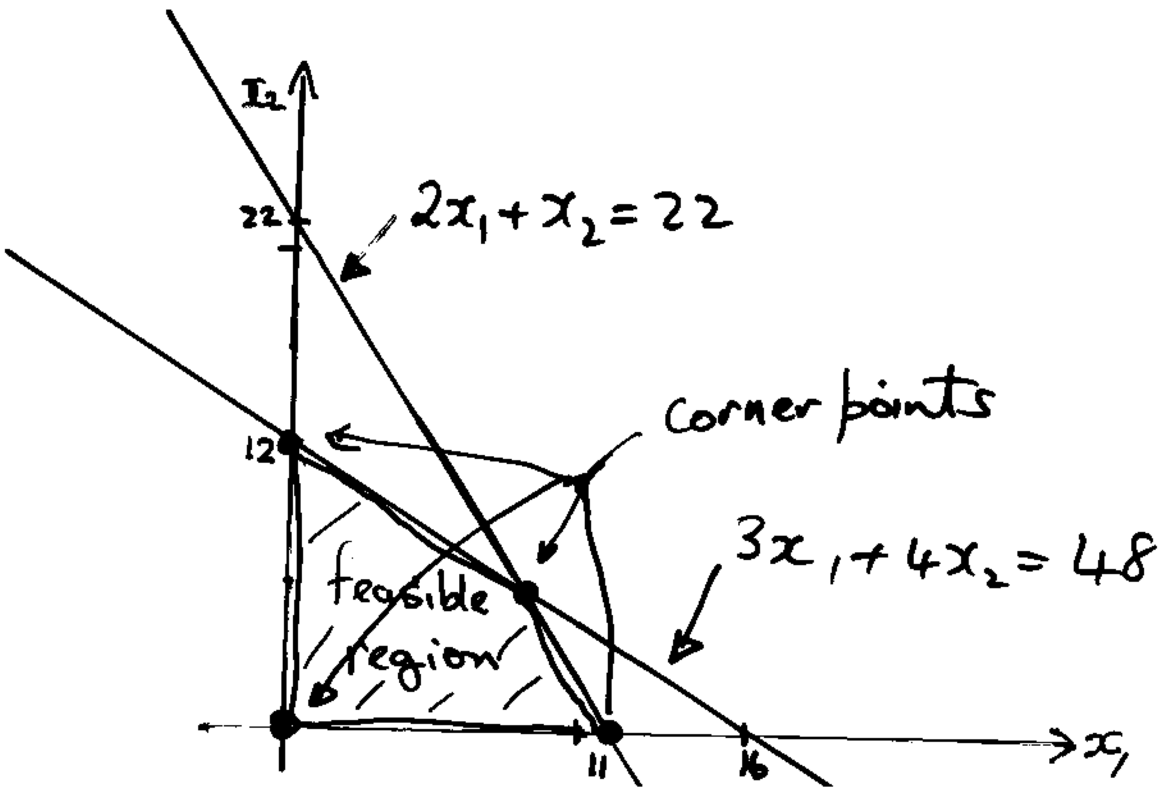
Maximise $f = 8x_1 + 4x_2$

subject to

$$3x_1 + 4x_2 \leq 48$$
$$2x_1 + x_2 \leq 22$$
$$x_1 \geq 0, \; x_2 \geq 0$$

Constraint lines (boundaries) are

$$3x_1 + 4x_2 = 48$$
$$2x_1 + x_2 = 22$$
$$x_1 = 0 \text{ and } x_2 = 0.$$



The corner points are

$$(x,y) = (0,0)\ ,\ (11,0)\ ,\ (0,12)\ ,\ (8,6)$$

At $(0,0)$ $f = 8x_1 + 4x_2 = 0$

At $(11,0)$ $f = 8 \times 11 = 88$

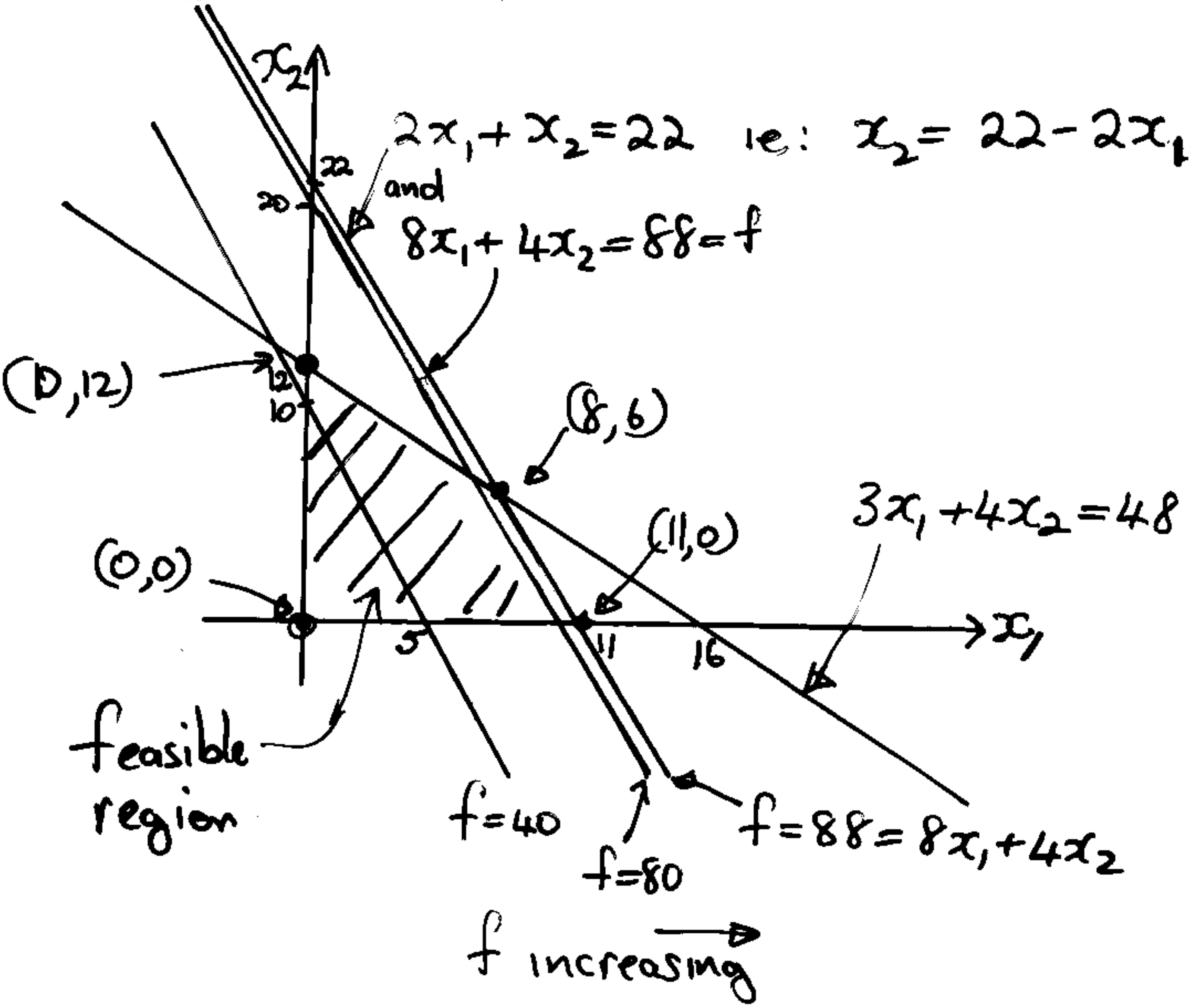
At $(0,12)$ $f = 4 \times 12 = 48$

At $(8,6)$ $f = 8 \times 8 + 6 \times 4 = 64 + 24 = 88$

So the maximum of f at corner point occurs at both $(x,y) = (11,0)$ and $(x,y) = (8,6)$

There is a 'tie'. This implies that we can choose any point on the line segment joining $(11,0)$ to $(8,6)$ and still get $f = 88$ eg Try $(x,y) = (10,2)$

This is because all constant objective function straight lines lie parallel to one of problem constraint boundaries.



By considering constant objective function straight lines with $f=40$, 80 and 88 one can argue that indeed our answer is the line segment joining $(8,6)$ and $(11,0)$ with $f=88$.

That is, our answer is

$$\text{Maximum of } f = 88 \text{ for } (x_1, x_2) = (t,\ 22-2t) \text{ with } 8 \le t \le 11.$$

The intersection of $3x_1 + 4x_2 = 48$

and $2x_1 + x_2 = 22$

$$\left[\begin{array}{cc|c} 2 & 1 & 22 \\ 3 & 4 & 48 \end{array}\right] \quad \tfrac{1}{2}R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & \tfrac{1}{2} & 11 \\ 3 & 4 & 48 \end{array}\right] \quad -3R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & \tfrac{1}{2} & 11 \\ 0 & 2\tfrac{1}{2} & 15 \end{array}\right] \quad \tfrac{2}{5}R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 1/2 & 11 \\ 0 & 1 & 6 \end{array}\right] \quad -\tfrac{1}{2}R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 6 \end{array}\right]$$

Hence $(x_1, x_2) = (8, 6)$

2 points : (8, 6) and (11, 0)

The straight line joining these is

$$y - 0 = m(x - 11)$$

slope

where $m = \dfrac{0-6}{11-8} = \dfrac{-6}{3} = -2$

So is $y = -2(x - 11)$

$= -2x + 22$

So letting $x = t$ then $y = 22 - 2t$

To give only points between (8, 6) and (11, 0) we restrict $8 \leq t \leq 11$ (ie: $8 \leq x \leq 11$)

Straight line equation $y - y_0 = m(x - x_0)$

slope

2001 exam

3. Solve graphically the following *non-standard* linear programming problem: Draw a graph with feasible region clearly marked and with all its corner points calculated. Write down all basic feasible solutions, and write down the values of (x, y) for which F takes its maximum value.

$$\text{Maximise} \qquad F = 51x + 34y$$

$$\text{subject to} \qquad \begin{aligned} 10x + 15y &\leq 120 \\ 9x + 6y &\leq 63 \\ 13x + 13y &\geq 65 \end{aligned}$$

with $x \geq 0$ and $y \geq 0$.

Answer: The correct graph
list of corner points
all basic feasible solutions
and

$F = 357$ is the maximum value of the objective function occuring at

$(x, y) = (t, 10\frac{1}{2} - \frac{3}{2}t)$ for $3 \leq t \leq 7$

2002 exam

3. Solve graphically the following *non-standard* linear programming problem: Draw a graph with the feasible region clearly marked, each of the boundary lines clearly marked and with all the corner points calculated. Include on your graph at least three 'constant objective function' straight lines, including one that goes through the points in the feasible region that give the maximum value of F. Using the graph of the feasible region and the three lines drawn give an argument for which values of (x, y) give the maximum value of F in the feasible region. Do not use the values of the objective function at ALL the corner points in your argument. You may use the values of the objective function at some corner points in your argument.

$$\text{Maximise} \qquad F = 10x + 10y$$

$$\text{subject to} \qquad \begin{aligned} 9x + 12y &\geq 36 \\ x \qquad &\leq 9 \\ 4x + 3y &\leq 48 \\ y &\leq 8 \end{aligned}$$

with $x \geq 0$ and $y \geq 0$.

Solution:

We can rewrite the problem as

Maximise $F = 10x + 10y$

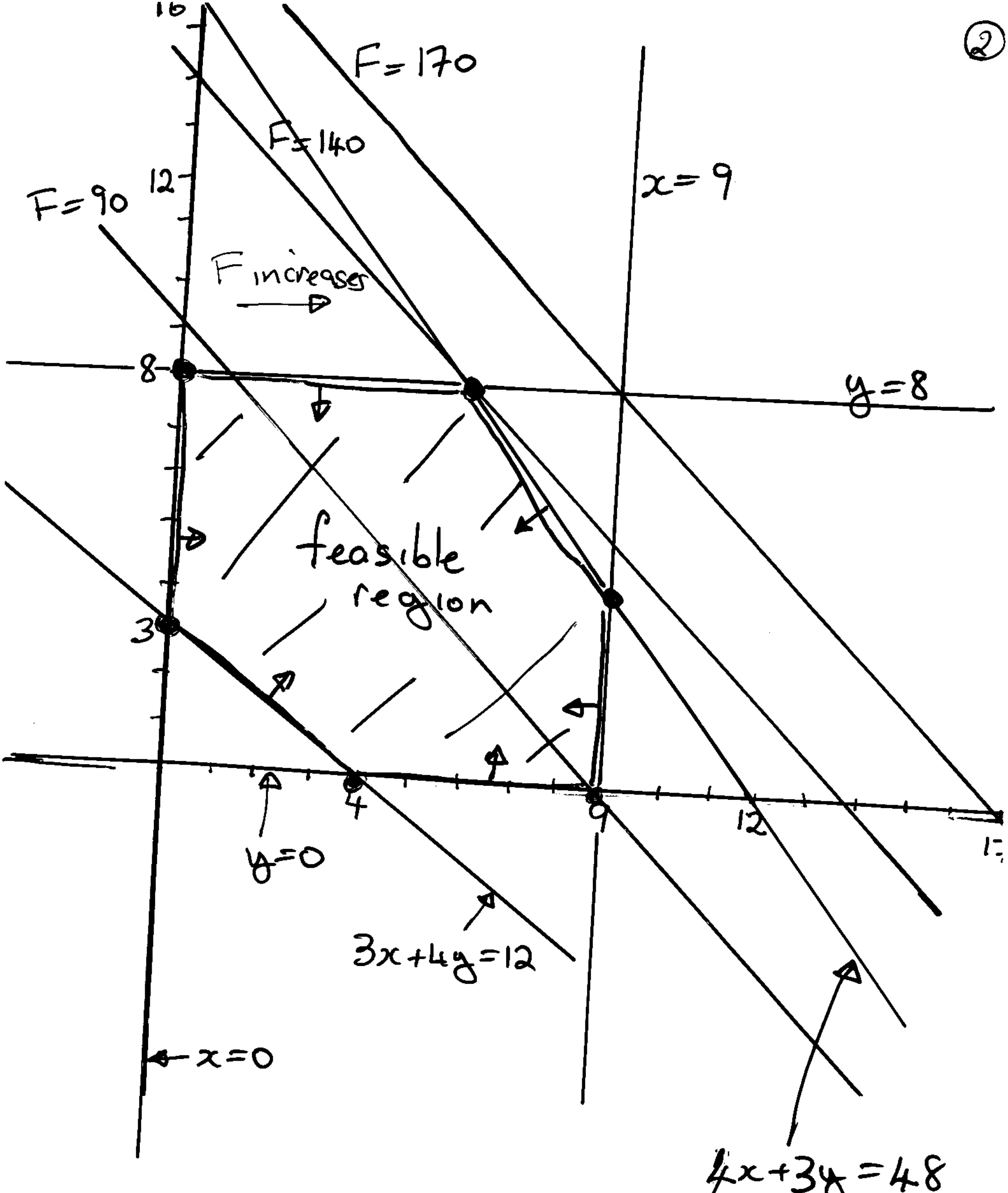
subject to
$$3x + 4y \geq 12$$
$$x \leq 9$$
$$y \leq 8$$
$$4x + 3y \leq 48$$

with $x \geq 0$ and $y \geq 0$.

The boundary lines are

$$3x + 4y = 12$$
$$4x + 3y = 48$$
$$x = 9$$
$$y = 8$$
$$x = 0$$
$$y = 0$$

②



● are 'corner points'

They are (0,3), (4,0), (9,0), (9,4), (6,8) and (0,8).

The value of F at the corner point (9,0) is

$F = 90$.

The value of F at the corner point (9,4) is

$F = 130$

The value of F at the corner point (6,8) is

$F = 140$.

At the point (9,8) which is out side the feasible region $F = 170$.

We have drawn the three constant objective function straight lines

$$F = 90 = 10x + 10y$$

$$F = 140 = 10x + 10y$$

and $F = 170 = 10x + 10y$

See diagram on previous page.

Clearly as F increases the graph of the objective function moves to the upper right. The graph of $F = 140$ cuts through the feasible region at only one point, the corner point $(6, 8)$.

Any value of F larger than $F = 140$ will lead to a straight line to the upper right of the line produced from $F = 140$ and hence must lie wholly outside the feasible region. Any other objective function line with points lying in the feasible region must be to the lower left of the line given by $F = 140$ and hence must be associated with a value less than 140. We hence conclude that $F = 140$ is the maximum that F can attain in the feasible region.

Hence the solution is

$$F = 140 \quad \text{at} \quad (x, y) = (6, 8)$$