Ex3. (Degeneracy)

Maximise $P = 5x_1 + 15x_2$

subject to $4x_1 + 2x_2 \leqslant 2000$

$x_1 + 3x_2 \leqslant 1500$

with $x_1 \geqslant 0$ and $x_2 \geqslant 0$.

Solution: We introduce slack variables s_1 and s_2 [redacted]

$$4x_1 + 2x_2 + s_1 = 2000$$

and $$x_1 + 3x_2 + s_2 = 1500$$

[redacted] constraints $s_1 \geqslant 0$ and $s_2 \geqslant 0$.

We rewrite the system as a [redacted] Tableaux with $x_1 = x_2 = 0$ as the [redacted] feasible solution (s_1, s_2 are basic)

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \end{array} \\ \left[\begin{array}{ccccc|c} 4 & 2 & 1 & 0 & 0 & 2000 \\ 1 & 3 & 0 & 1 & 0 & 1500 \\ \hline -5 & -15 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \text{Quotients} \\ \frac{2000}{2} = 1000 \\ \frac{1500}{3} = 500 \\ \frac{1}{3}R_2 \Rightarrow R_2 \end{array}$$

↑ pivot column

$$\sim \left[\begin{array}{ccccc|c} 4 & 2 & 1 & 0 & 0 & 2000 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 500 \\ \hline -5 & -15 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} -2R_2+R_1 \to R_1 \\ \\ 15R_1+R_3 \to R_3 \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & RHS \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} \frac{10}{3} & 0 & 1 & -\frac{2}{3} & 0 & 1000 \\ \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 500 \\ \hline 0 & 0 & 0 & 5 & 1 & 7500 \end{array}\right]$$

Algorithm STOP!

The zero we deliberately obtained.

A 'fluke' zero

Interpretation: This fluke zero is a symptom of Degeneracy. That is, a zero coefficient of a non-basic variable in the objective function equation.

Degeneracy means there is more than one point (x_1, x_2) that gives our maximum value 7500 in this case. How? ...

Since there are no negative values in the bottom row we know that one maximum solution is given by

$P = 7500$ when $x_1 = 0$, $x_2 = 500$

with slack variables $S_1 = 1000$, $S_2 = 0$

But let us examine the equation associated with the bottom row:

$$P = 7500 - 5 S_2$$

Certainly $P = 7500$ if $S_2 = 0$ but we have also specified $x_1 = 0$ → This clearly is not necessary → we can change x_1 keeping $S_2 = 0$ without affecting P.

Our Tableaux says that

$$\frac{10}{3} x_1 + S_1 - \frac{2}{3} S_2 = 1000$$

and $\frac{1}{3} x_1 + x_2 + \frac{1}{3} S_2 = 500$

Since we must have $S_2 = 0$ for $P = 7500$ we actually have that

$$\left.\begin{aligned} \frac{10}{3} x_1 + S_1 &= 1000 \\ \frac{1}{3} x_1 + x_2 &= 500 \end{aligned}\right\} (\Delta)$$

So any values of x_1, x_2 and S_1 satisfying these equations with the constraints that x_1, x_2 and $S_1 \geqslant 0$ gives us $P = 7500$

The equations (Δ) are an underdetermined set:

Let $x_1 = t$ (a parameter)

Then $S_1 = 1000 - \frac{10}{3} t$

and $x_2 = 500 - \frac{1}{3} t$

Since $x_1 \geqslant 0$ then $t \geqslant 0$

Since $x_2 \geqslant 0$ then $500 - \frac{1}{3} t \geqslant 0$

so $t \leqslant 1500$

Since $S_1 \geq 0$ Then $1000 - \frac{10}{3}t \geq 0$

so $t \leq 300$.

Since all 3 constraints need To be satisfied we can Take only t values obeying

$$0 \leq t \leq 300$$

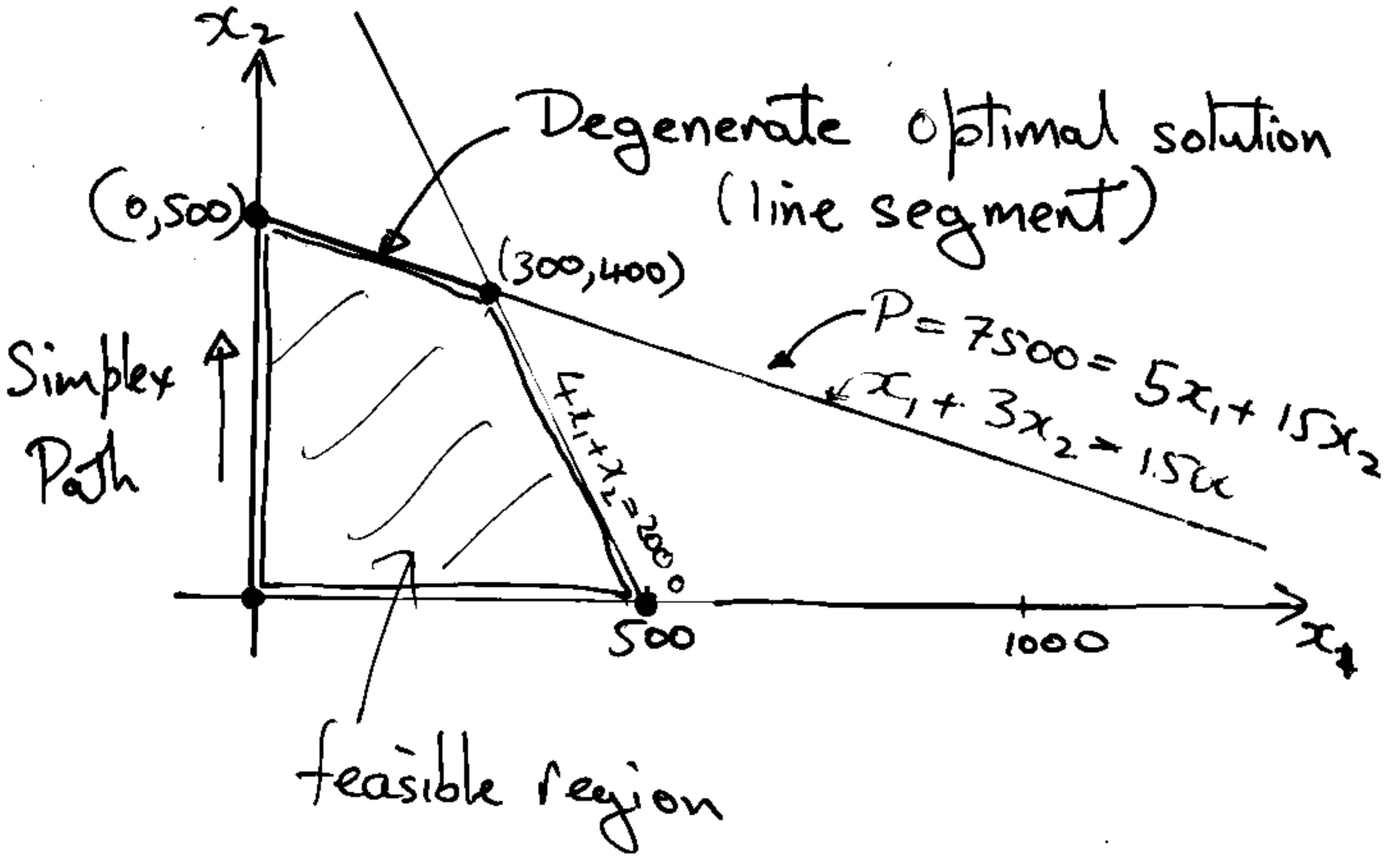
Hence our complete solution To the LP problem is:

$$\boxed{\begin{array}{l}\text{The maximum of } P \text{ is } 7500 \\ \text{when } (x_1, x_2) = (t, 500 - \frac{t}{3}) \\ \text{for } 0 \leq t \leq 300.\end{array}}$$

Note That $S_1 = 1000 - \frac{10}{3}t$ for $0 \leq t \leq 300$

and $S_2 = 0$.

We should check our solution by substitution back into original inequations etc.



Note that when $t=0$ $(x_1, x_2) = (0, 500)$

and when $t=300$ $(x_1, x_2) = (300, 400)$

both are basic feasible solutions.

But in higher dimension degenerate answers can be sections of planes etc...

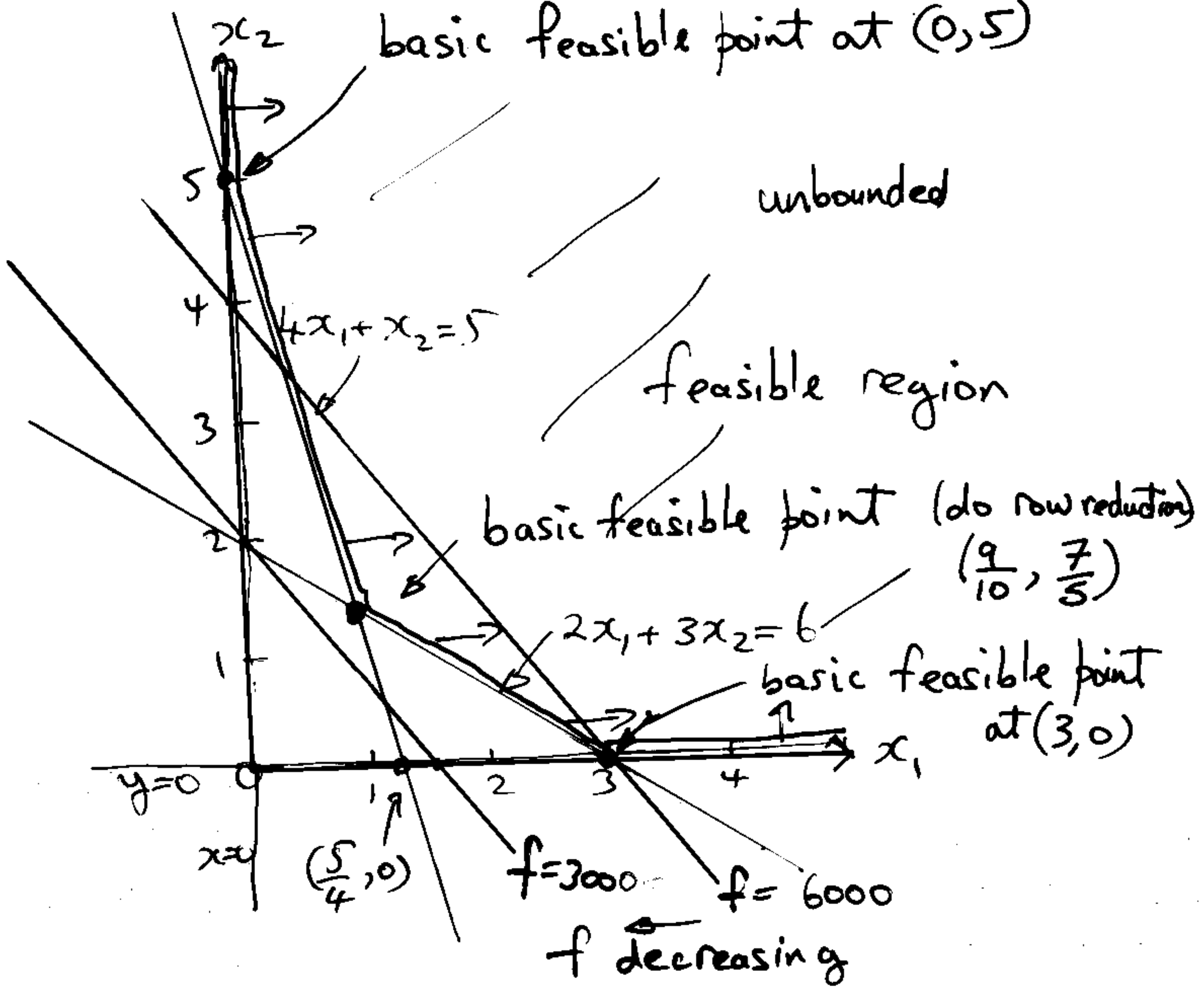
The Standard Minimum Problem

BZB 5-5

Solve graphically:

Minimise $f = 2000x_1 + 1500x_2$

subject to
$$4x_1 + x_2 \geq 5$$
$$2x_1 + 3x_2 \geq 6 \qquad (\mathrm{I})$$
$$x_1, x_2 \geq 0$$



Since decreasing the value of f results in an objective function straight line to the left of another we conclude that there exists a solution to this problem at one of the basic feasible points. Since $f = 6000$ line runs through the basic feasible point $(3, 0)$ and there is only one basic feasible point to the left of that objective function line the minimum value of f must occur at the basic feasible point $\left(\frac{9}{10}, \frac{7}{5}\right)$

At $(x_1, x_2) = \left(\frac{9}{10}, \frac{7}{5}\right)$ $\quad f = 2000 \times \frac{9}{10} + 1500 \times \frac{7}{5}$

$$= 3900$$

So Minimum of $f = 3900$ at $x_1 = \frac{9}{10}$, $x_2 = \frac{7}{5}$

Something looks familiar?

The solution to:

$$\boxed{\begin{array}{ll} \text{Maximise} & g = 5y_1 + 6y_2 \\ \text{subject to} & 4y_1 + 2y_2 \le 2000 \\ & y_1 + 3y_2 \le 1500 \\ & y_1 \ge 0,\ y_2 \ge 0 \end{array}}$$ (II)

is Maximum of $g = 3900$

occurs at $(y_1, y_2) = (300, 400)$

with slack $(0,0)$

compared to solution (I)

Minimum of $f = 3900$ at $(x_1, x_2) = \left(\frac{9}{10}, \frac{7}{5}\right)$

with slack $(0,0)$

Look at problems I & II → compare coefficients

... This is <u>not</u> a coincidence

⋮

$\boxed{\text{This is DUALITY}}$

The final Tableaux of The solution To (II) is

$$\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & s_1 & s_2 & g & RHS \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 0 & 3/10 & -1/5 & 0 & 300 \\
0 & 1 & -1/10 & 2/5 & 0 & 400 \\
\hline
0 & 0 & \frac{9}{10} & \frac{7}{5} & 1 & 3900
\end{array}\right]
\end{array}$$

Hang on a minute ?

<u>Duality idea</u>

Problem (II) is called The dual of (I).
To solve (I) we can write down The dual problem II and solve that instead – we can Then read off The solution To (I) from the final Tableaux of the solution To (II).
In This way we get The solutions To Two LP problems at once.
The proof of The <u>dual Theorem</u> is not simple and depends on matrix algebra.

The dual problem: general case

The dual problem to

$$\boxed{\begin{aligned} &\text{Min } f = \underset{\sim}{c}^T \underset{\sim}{x} \\ &\text{subject to } \quad B\underset{\sim}{x} \geq \underset{\sim}{b} \\ &\text{and} \quad \underset{\sim}{x} \geq 0 \end{aligned}}$$

is

$$\boxed{\begin{aligned} &\text{Max } g = \underset{\sim}{b}^T \underset{\sim}{y} \\ &\text{subject to } \quad B^T\underset{\sim}{y} \leq \underset{\sim}{c} \\ &\text{and} \quad \underset{\sim}{y} \geq 0 \end{aligned}}$$

and vice versa.

Note: For the maximisation problem to be standard we need $\underset{\sim}{c} \geq 0$ ie $\underset{\sim}{c}^T \geq 0^T$
On the other hand the elements of $\underset{\sim}{b}$ can be of any sign. The same is true for the minimisation problem: $\underset{\sim}{c}^T \geq 0^T$.

Writing down The dual problem:

Ex 1. Minimise $f = 5x_1 + 2x_2$

subject To $4x_1 + 2x_2 \geqslant 38$

$3x_1 + 2x_2 \geqslant 32$

$x_1 + 6x_2 \geqslant -43$

$x_1 \geqslant 0, \; x_2 \geqslant 0.$

So $\underset{\sim}{c}^T = [5\; 2]$, $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 38 \\ 32 \\ -43 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 2 \\ 3 & 2 \\ 1 & 6 \end{bmatrix}$

and so $\underset{\sim}{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $\underset{\sim}{b}^T = [38\; 32\; -43]$

$B^T = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 6 \end{bmatrix}$

Hence The dual problem is

Maximise $g = 38y_1 + 32y_2 - 43y_3$

subject To $4y_1 + 3y_2 + y_2 \leq 5$

$2y_1 + 2y_2 + 6y_3 \leq 2$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$.

Ex2. Minimise $f = 8x_1 + 10x_2 + x_3$

subject To $2x_1 - 2x_2 + x_3 \geq 17$

$x_1 + x_2 - x_3 \geq 12$

$x_1, x_2, x_3 \geq 0$.

The dual is

Maximise $g = 17x_1 + 12x_2$

subject To $2y_1 + y_2 \leq 8$

$-2y_1 + y_2 \leq 10$

$y_1 - y_2 \leq 1$

$y_1 \geq 0,\ y_2 \geq 0$.

Reading off the answer for a dual problem — by example.

The definitive example

The original problem

$$\text{Minimise} \quad f = 10x_1 + 30x_2$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 &\geq 16 \\ x_1 + x_2 &\geq 12 \\ x_1 + 2x_2 &\geq 14 \end{aligned}$$

with $x_1 \geq 0$ and $x_2 \geq 0$.

The dual problem is

$$\text{Maximise} \quad g = 16y_1 + 12y_2 + 14y_3$$

$$\text{subject to} \quad \begin{aligned} 2y_1 + y_2 + y_3 &\leq 10 \\ y_1 + y_2 + y_2 &\leq 30 \end{aligned}$$

with $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$

The final Tableaux was

y_3 enters, y_2 exits

$$\left[\begin{array}{cccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 10 \\ -3 & -1 & 0 & -2 & 1 & 0 & 10 \\ \hline 12 & 20 & & 14 & 0 & 1 & 140 \end{array}\right]$$

y_3, S_2 are basic variables

Slack for minimum problem

Solution point for minimum problem

Solution to the maximum problem is

$g = 140$ is the maximum value

achieved at $(y_1, y_2, y_3) = (0, 0, 10)$

with slack $(S_1, S_2) = (0, 10)$

Solution to the minimum problem is

$f = 140$ is the minimum value

when $(x_1, x_2) = (14, 0)$

with slack $(t_1, t_2, t_3) = (12, 2, 0)$

where the slack variables are defined as

$$2x_1 + x_2 - t_1 = 16$$
$$x_1 + x_2 - t_2 = 12$$
$$x_1 + 2x_2 - t_3 = 14$$

$$
\begin{array}{c|cccccc|c|c}
 & y_1 & y_2 & y_3 & S_1 & S_2 & g & \text{RHS} & \text{Quotient} \\
\hline
\rightarrow & 2 & 1 & 1 & 1 & 0 & 0 & 10 & 5 \\
 & 1 & 1 & 2 & 0 & 1 & 0 & 30 & 30 \\
\hline
 & -16 & -12 & -14 & 0 & 0 & 1 & 0 & \\
 & \uparrow & & & & & & &
\end{array}
$$

S_1, S_2 basic

$\frac{1}{2}R_1 \to R_1$

S_1 exits
y_1 enters

$$
\sim \left[\begin{array}{cccccc|c}
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\
1 & 1 & 2 & 0 & 1 & 0 & 30 \\
\hline
-16 & -12 & -14 & 0 & 0 & 1 & 0
\end{array}\right]
$$

$-R_1 + R_2 \to R_2$

$16R_1 + R_3 \to R_3$

$$
\begin{array}{c}
\rightarrow \\ \sim \\ \\
\end{array}
\left[\begin{array}{cccccc|c}
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\
0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 1 & 0 & 25 \\
\hline
0 & -4 & -6 & 8 & 0 & 1 & 80 \\
 & \uparrow & & & & &
\end{array}\right]
\begin{array}{c}
10 \\ 50 \\ \\
\end{array}
$$

y_1, S_2 basic

$2R_1 \to R_1$

y_1 exits
y_2 enters

$$
\sim \left[\begin{array}{cccccc|c}
2 & 1 & 1 & 1 & 0 & 0 & 10 \\
0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 1 & 0 & 25 \\
\hline
0 & -4 & -6 & 8 & 0 & 1 & 80
\end{array}\right]
$$

$-\frac{1}{2}R_1 + R_2 \to R_2$

$4R_1 + R_3 \to R_3$

$$
\begin{array}{c}
\rightarrow \\ \sim \\ \\
\end{array}
\left[\begin{array}{cccccc|c}
2 & 1 & 1 & 1 & 0 & 0 & 10 \\
-1 & 0 & 1 & -1 & 1 & 0 & 20 \\
\hline
8 & 0 & -2 & 12 & 0 & 1 & 120 \\
 & & \uparrow & & & &
\end{array}\right]
\begin{array}{c}
10 \\ 20 \\ \\
\end{array}
$$

no choice

y_2, S_2 basic

$-R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

4. Use the simplex method to solve the following standard maximum problem.

$$\text{Maximise} \qquad g = 12y_2 + 13y_3$$

$$\text{subject to} \qquad \begin{aligned} 2y_1 + 2y_2 &\le 6 \\ 12y_1 + 6y_2 + 12y_3 &\le 24 \\ 15y_1 - 16y_3 &\le 8 \end{aligned}$$

with $y_1 \geq 0$, $y_2 \geq 0$ and $y_3 \geq 0$.

At each step show clearly the row operation(s) that you perform and clearly circle the pivot element. Inspect your final tableau and state the maximum possible value of g and all the values of (y_1, y_2, y_3) for which this maximum occurs.

We introduce slack variables s_1, s_2, s_3 so that the problem constraints become

$$2y_1 + 2y_2 + s_1 = 6,$$

$$12y_1 + 6y_2 + 12y_3 + s_2 = 24,$$

and $$15y_1 - 16y_3 + s_3 = 8,$$

where $s_1 \geq 0$, $s_2 \geq 0$, and $s_3 \geq 0$.

We rewrite the system as a simplex Tableaux

$$\begin{array}{c}
\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & P & \text{RHS} \end{array} \\
\left[\begin{array}{ccccccc|c}
2 & 2 & 0 & 1 & 0 & 0 & 0 & 6 \\
12 & 6 & 12 & 0 & 1 & 0 & 0 & 24 \\
15 & 0 & -16 & 0 & 0 & 1 & 0 & 8 \\
\hline
0 & -12 & -13 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\quad
\begin{array}{c} \text{Quotients} \\ — \\ 2 = \frac{24}{12} \\ — \\ \end{array}$$

pivot column

$\frac{1}{12}R_2 \to R_2$

Pivot element is circled.

$$\left[\begin{array}{ccccccc|c} 2 & 2 & 0 & 1 & 0 & 0 & 0 & 6 \\ 1 & \frac{1}{2} & 1 & 0 & \frac{1}{12} & 0 & 0 & 2 \\ 15 & 0 & -16 & 0 & 0 & 1 & 0 & 8 \\ \hline 0 & -12 & -13 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ 16R_2+R_3 \to R_3 \\ 12R_2+R_4 \to R_4 \end{array}$$

$$\begin{array}{c} \left[\begin{array}{ccccccc|c} 2 & 2 & 0 & 1 & 0 & 0 & 0 & 6 \\ 1 & \frac{1}{2} & 1 & 0 & \frac{1}{12} & 0 & 0 & 2 \\ 31 & 8 & 0 & 0 & \frac{4}{3} & 1 & 0 & 40 \\ \hline 13 & -\frac{11}{2} & 0 & 0 & \frac{13}{12} & 0 & 1 & 26 \end{array}\right] \\ \begin{array}{l} \qquad\quad \uparrow \text{pivot column} \end{array} \end{array}$$

Quotients

$\frac{6}{2} = 3$

$2/\tfrac{1}{2} = 4$

$40/8 = 5$

$\frac{1}{2}R_1 \to R_1$

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 3 \\ 1 & \frac{1}{2} & 1 & 0 & \frac{1}{12} & 0 & 0 & 2 \\ 31 & 8 & 0 & 0 & \frac{4}{3} & 1 & 0 & 40 \\ \hline 13 & -\frac{11}{2} & 0 & 0 & \frac{13}{12} & 0 & 1 & 26 \end{array}\right] \quad \begin{array}{l} \\ -\frac{1}{2}R_1+R_2 \to R_2 \\ -8R_1+R_3 \to R_3 \\ \frac{11}{2}R_1+R_4 \to R_4 \end{array}$$

$$\sim \left[\begin{array}{ccccccc|c} 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 3 \\ \frac{1}{2} & 0 & 1 & -\frac{1}{4} & \frac{1}{12} & 0 & 0 & \frac{1}{2} \\ 23 & 0 & 0 & -4 & \frac{4}{3} & 1 & 0 & 16 \\ \hline \frac{37}{2} & 0 & 0 & \frac{11}{4} & \frac{13}{12} & 0 & 1 & \frac{85}{2} \end{array}\right]$$

STOP since there are no negative elements in bottom row.

We deduce that the maximum value of g is $\frac{85}{2}$ occurring at $(y_1, y_2, y_3) = (0, 3, \frac{1}{2})$.

The slack variables at this point have values $(s_1, s_2, s_3) = (0, 0, 16)$.

Answer check: (We note that $y_1, y_2, y_3 \geq 0$

If we label at $(y_1, y_2, y_3) = (0, 3, \frac{1}{2})$).

$$2y_1 + 2y_2 \leq 6 \quad \text{(i)}$$

$$12y_1 + 6y_2 + 12y_3 \leq 24 \quad \text{(ii)}$$

$$15y_1 - 16y_3 \leq 8 \quad \text{(iii)}$$

We note that at $(y_1, y_2, y_3) = (0, 3, \frac{1}{2})$

LHS of (i) $= 2\times 0 + 2\times 3 = 6 \leq 6 =$ RHS of (i)

(and $S_1 = 0$) so OK ✓

LHS of (ii) $= 12\times 0 + 6\times 3 + 12\times\frac{1}{2}$

$= 18 + 6 = 24 \leq 24 =$ RHS of (ii) ✓

(and $S_2 = 0$) so OK

LHS of (iii) $= 15\times 0 - 16\times\frac{1}{2} = -8 \leq 8 =$ RHS of (iii) ✓

so OK.

(and $S_3 = 8 - (-8) = 16$).

Now $g = 12y_2 + 13y_3 = 12\times 3 + 13\times\frac{1}{2}$

$= 36 + \frac{13}{2} = \frac{72+13}{2} = \frac{85}{2}$

so OK. ✓