Topic 7 — Introduction to differential equations

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Differential equations

Slope fields

Recap

Euler method

The improved Euler method
Previously . . .

In Topic 6 we had

\[ y = \frac{1}{x} \quad \text{giving} \quad \frac{dy}{dx} = -\frac{1}{x^2} \]

\[ xy = 1 \quad \longrightarrow \quad \frac{dy}{dx} = -\frac{y}{x} \]

\[ x^2 + y^2 = 1 \quad \longrightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \]

\[ x^3 + y + xe^y = 1 \quad \longrightarrow \quad \frac{dy}{dx} = -\frac{3x^2 + e^y}{1 + xe^y} \]

- Can we go backwards?
- Leads us to “Differential Equations” or DEs.
Introduction to DEs

- See BZB Chapter 14.
- Some terminology:
  - Differential equation
  - Slope field
  - Order
  - Initial condition
  - General solution
  - Particular solution
  - Implicit solution
- We do mostly “linear” and “separable” DEs.
Definition

A first order differential equation for an unknown function $y(x)$ involves its first derivative $\frac{dy}{dx}$.

- In general $g \left( x, y, \frac{dy}{dx} \right) = 0$

- but usually $\frac{dy}{dx} = h(x, y)$.

- We want to find $y(x)$ as either
  - an explicit function $y = y(x)$, or
  - an implicit function given by $F(x, y(x)) = 0$.

- We will also write $y'$ instead of $\frac{dy}{dx}$.

- We will only consider $x, y \in \mathbb{R}$ — real numbers.
Examples of first order DEs

<table>
<thead>
<tr>
<th>Standard</th>
<th>Alternative notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \frac{dy}{dx} = 3xy )</td>
<td>( y' = 3xy )</td>
</tr>
<tr>
<td>(b) ( x^2 \frac{dy}{dx} = y^2 + y + 2 )</td>
<td>( y' = y^2 + y + 2 )</td>
</tr>
<tr>
<td>(c) ( \frac{dy}{dx} + 2xy = x^4 )</td>
<td>( y' + 2xy = x^4 )</td>
</tr>
<tr>
<td>(d) ( \frac{dy}{dx} = -\frac{x}{y} )</td>
<td>( y' = -\frac{x}{y} )</td>
</tr>
<tr>
<td>(e) ( \frac{dy}{dx} = 3x^2 )</td>
<td>( y' = 3x^2 )</td>
</tr>
</tbody>
</table>
Finding $y(x)$

What is a solution to a DE?

- Let us suppose we have some explicit expression for $y(x)$ we think is a solution.
- Substitute it into the DE
- If the DE becomes an identity (such as $x^3 = x^3$) then we have a solution!
An example

- \( y = e^{x^2} \) is one solution of the DE \( \frac{dy}{dx} = 2xy \).

- Check LHS:

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{de^u}{du} 2x = 2xe^u = 2xe^{x^2}
\]

- Check RHS: \( 2xy = 2xe^{x^2} \)

- So LHS = RHS \( \checkmark \).

- Note that \( y = e^{x^2+5} \) also gives \( \frac{dy}{dx} = 2xy \).
Another example

- $y = x - 1 + 3e^{-x}$ is one solution of
  $$\frac{dy}{dx} = x - y$$

- Check LHS: $\frac{dy}{dx} = 1 - 3e^{-x}$.
- Check RHS:
  $$x - y = x - (x - 1 + 3e^{-x})$$
  $$= 1 - 3e^{-x}$$

- So LHS = RHS $\checkmark$. 
Numerical solutions

► Ideally we would like to find a “nice” solution for $y(x)$ — either an explicit or implicit expression.
► Often it is too hard to find one (if it exists at all).
► Often we don’t need it!
   — If we are modelling “something” we are more interested in “something” than nice solutions.
► Instead look for an approximate or numerical solution:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>1.8145</td>
</tr>
<tr>
<td>0.2</td>
<td>1.6562</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5225</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

a big table of numbers

or an approximate plot of the “solution curve”.
Slope fields

- Consider the DE $\frac{dy}{dx} = x - y$.
- The equation defines the slope of the function $y(x)$ at any point $(x, y)$.
- If a “solution curve” passes through a point $(x, y)$ we can calculate its slope at that point.
  - At $(1, 1)$ the slope is 0.
  - At $(1, 2)$ the slope is $-1$.
- We can get a feel for solutions by plotting the “slope field”.
- Given a point $(x, y)$ draw a short at that point with slope given by the DE.
- Do this for many points in the plane.
Slope field

For $\frac{dy}{dx} = x - y$ this gives:
Slope field — solution curves

Draw some solution curves

- $y(1) = 0 \rightarrow x - 1 + 2e^{-x}$ top
- $y(0) = 0 \rightarrow x - 1 + e^{-x}$ middle
- $y(-1) = 0 \rightarrow x - 1$ bottom
- General solution $y(x) = x - 1 + Ae^{-x}$ with $A \in \mathbb{R}$.
Another slope field example

\[ \frac{dy}{dx} = y \]

- General solution is \( y = Ae^x \)
- The constant \( A \) is arbitrary and \( A \in \mathbb{R} \).
- Normally first order DE \( \rightarrow 1 \) constant.
- \( y = \frac{1}{2}e^x \) is a particular solution.
  Satisfies initial condition \( y(0) = 1/2 \) or \( y(1) = e/2 \).
Solution family

\[
\frac{dy}{dx} = x - y
\]

- Some solutions are
  - \( y = x - 1 + 3e^{-x} \)
  - \( y = x - 1 + 234.08123e^{-x} \)
  - \( y = x - 1 \)
- Any function \( y = x - 1 + Ae^{-x}, \ A \in \mathbb{R} \) is a solution.
  - LHS: \( \frac{d}{dx} (x - 1 + Ae^{-x}) = 1 - Ae^{-x} \).
  - RHS: \( x - (x - 1 + Ae^{-x}) = 1 - Ae^{-x} \).
  - So LHS = RHS √.
Solution families

- $y = x - 1 + Ae^{-x}, A \in \mathbb{R}$ is the “general solution” of the DE.
- All these $y(x)$ are the “solution family” of curves (an infinite number of curves).
- A “singular solution” is has a different form to the general solution.
- Eg: $y = Cx - \frac{1}{4}C^2$ is a solution family for $y = xy' - (y')^2$.
- But $y = x^2$ is also a solution — a singular solution.
Recap

A DE: for example $\frac{dx}{dy} = x - y$

- Valid for all $x$ and $y$.
- To get a global picture of the DE use a slope field.
- Many solutions.

An initial value problem:

$$\text{IVP} = \begin{cases} \frac{dy}{dx} = x - y & \text{a DE} \\ y(0) = 1 & \text{initial condition} \end{cases}$$

- Has one solution — a function $y(x)$ (possibly implicit).
- Picture of solution goes through the initial condition and “agrees” with the slope field.
- At any point the slope of the solution is given by the slope field.
Towards the idea

- The solution curve must “agree” with the slope field.
- Hence we can use the slope field to get a picture of the solution.

- Better than that — we can use the slope field to get a numerical solution.
- Start at the initial condition and take short steps in the direction of the slope field.
- This is the beginning of the Euler method.
Euler method

- We start with an initial value problem:
  - DE: \( \frac{dy}{dx} = f(x, y). \)
  - Initial condition \( y(x_0) = y_0. \)

and require a numerical method for constructing an approximate solution.

- The solution is a curve going through \((x_0, y_0)\).
Idea of Euler method

- Start at a point \((x_0, y_0)\) and find tangent line.
- Use the tangent line as an approximate solution over a short distance \(\Delta x\).
- Move along tangent line to new point \((x_1, y_1)\).
- Repeat.
Euler procedure

1. Calculate slope of (unknown) solution curve at $(x_0, y_0)$ from DE.
2. Hence find equation to tangent line to curve at $(x_0, y_0)$.
3. From tangent line equation find new $y$-value at the $x$-value $x_1 = x_0 + \Delta x$.
4. This gives new point $(x_1, y_1)$.
5. Repeat steps 1 to 5.
Finding formulas

- \( P(x_0, y_0) \) is on the solution curve \( y = y(x) \).
- \( Q(x_1, y_1) \) is “close” to the solution curve.
- The “slope field” at \( P \) is \( \left. \frac{dy}{dx} \right|_{(x_0, y_0)} = f(x_0, y_0) \).
- So the gradient of the tangent line is \( f(x_0, y_0) \)
- The equation is \( (y - y_0) = f(x_0, y_0)(x - x_0) \).
Finding formulas

- The point $Q(x_1, y_1)$ is on the tangent line so
  \[
  (y_1 - y_0) = f(x_0, y_0)(x_1 - x_0) \quad \text{rearrange a bit}
  \]
  \[
  y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) \quad \text{a bit more \ldots}
  \]
  \[
  y_1 = y_0 + f(x_0, y_0)\Delta x
  \]

- We started at $(x_0, y_0)$ as our initial point on the solution curve and we have just found our next point $(x_1, y_1)$ as
  \[
  x_1 = x_0 + \Delta x
  \]
  \[
  y_1 = y_0 + f(x_0, y_0)\Delta x
  \]
Finding formulas

- Repeating this gives another point \((x_2, y_2)\) as

\[
\begin{align*}
x_2 &= x_1 + \Delta x \\
y_2 &= y_1 + f(x_1, y_1)\Delta x
\end{align*}
\]

and so on...

- In general the \((n + 1)^{th}\) point is given in terms of the \(n^{th}\) point as

\[
\begin{align*}
x_{n+1} &= x_n + \Delta x \\
y_{n+1} &= y_n + f(x_n, y_n)\Delta x
\end{align*}
\]
Euler method — an example

Solve approximately

\[
\left\{ \begin{array}{l}
\frac{dy}{dx} = x + y \\
y(0) = 0
\end{array} \right.
\]

using the Euler method with \( \Delta x = 0.2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( f(x_n, y_n) = \frac{x_n + y_n}{x_n + y_n} )</th>
<th>( x_{n+1} = x_n + \Delta x )</th>
<th>( y_{n+1} = y_n + f(x_n, y_n)\Delta x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.2</td>
<td>0.0 + 0.0 \times 0.2 = 0.0</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.0 + 0.2 \times 0.2 = 0.04</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.04</td>
<td>0.44</td>
<td>0.6</td>
<td>0.04 + 0.44 \times 0.2 = 0.128</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>0.128</td>
<td>0.728</td>
<td>0.8</td>
<td>0.128 + 0.728 \times 0.2 = 0.2736</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>0.2736</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Current point | Next point
Improving the approximation

- If the slope field is changing quickly then Euler may not be very accurate.
- The “improved Euler method” is a simple but effective improvement.

1. Start at \((x_0, y_0)\).
2. Do a single Euler iteration to find \((x_1, y_1^*)\).
3. Go back to \((x_0, y_0)\) and use the average of the slopes at \((x_0, y_0)\) and \((x_1, y_1^*)\) to find a new point \((x_1, y_1)\). Discard \((x_1, y_1^*)\).
4. Repeat.

- The Euler point \((x_1, y_1^*)\) is a temporary intermediate point only.
A picture

Outline
Differential equations
Slope fields
Recap
Euler method
Improved Euler
The idea
Finding formulas
An example
Notes

A tangent line at point $(x_0, y_0)$ is drawn, and an intermediate point is indicated at $x_1$. The difference $\Delta x = x_1 - x_0$ is shown on the x-axis.
Improved Euler: The idea

Finding formulas

An example

Notes
Finding formulas

- Starting point \((x_0, y_0)\)
- Intermediate point \((x_1, y_1^*)\)
  
  where \[
  \begin{align*}
  x_1 &= x_0 + \Delta x \\
  y_1^* &= y_0 + f(x_0, y_0)\Delta x
  \end{align*}
  \]
- Averaged slope at \((x_0, y_0)\) and \((x_1, y_1^*)\) is
  \[
  m = \frac{1}{2} \left( f(x_0, y_0) + f(x_1, y_1^*) \right)
  \]
- The new point \((x_1, y_1)\) is therefore given by
  \[
  \begin{align*}
  x_1 &= x_0 + \Delta x \\
  y_1 &= y_0 + m\Delta x
  \end{align*}
  \]
Finding formulas

- In general the next point \((x_{n+1}, y_{n+1})\) in the improved Euler method is given by

\[
\begin{align*}
x_{n+1} & = x_n + \Delta x \\
y_{n+1} & = y_n + m\Delta x
\end{align*}
\]

- Where \(m\) is the average slope at \((x_n, y_n)\) and \((x_{n+1}, y_{n+1})\).

\[
m = \frac{1}{2} \left( f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*) \right)
\]

\[
y^* = y_n + f(x_n, y_n)\Delta x
\]

- Warning: \((x_{n+1}, y_{n+1}^*)\) is not the \((n+1)^{th}\) Euler point. It is the Euler point obtained from the \(n^{th}\) Improved Euler point.
Improved Euler method — an example

Solve approximately

\[
\begin{cases}
\frac{dy}{dx} = x + y \\
y(0) = 0
\end{cases}
\]

using the Improved Euler method with \( \Delta x = 0.2 \).

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<th>( y_n )</th>
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<th>( x_{n+1} )</th>
<th>( y_{n+1}^* )</th>
<th>( f(x_{n+1}, y_{n+1}^*) )</th>
<th>( m )</th>
<th>( y_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0.0</td>
<td>0.0</td>
<td>0.2</td>
<td>0.0</td>
<td>0.2</td>
<td>0.1</td>
<td>0.02</td>
</tr>
<tr>
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<td>0.064</td>
<td>0.464</td>
<td>0.342</td>
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</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.0884</td>
<td>Current</td>
<td>Intermed.</td>
<td></td>
<td></td>
<td></td>
<td>Next</td>
</tr>
</tbody>
</table>
Improved Euler much better than Euler.

Still, with either, we don’t know how good the answer is.

We cannot answer the important question: “How small do we have to make $\Delta x$ to achieve a certain accuracy?”