Topic 8 — Taylor polynomials and series

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Introduction

Taylor polynomials

Taylor series

Solutions to DEs
Introduction

- Taylor polynomials — approximation to a function near some point.
- Taylor series — exact representation of a function.

- Not in BZB, but try any standard calculus text such as
  - *Calculus* by Thomas and Finney.
  - *Calculus* by Grossman
Higher derivatives — quick review

Consider the function $y = (1 + x)e^{-x}$. Derivatives are $y' = -xe^{-x}$ and $y'' = (x - 1)e^{-x}$.
Higher derivatives — quick review

- $y'$ tells us about the rate of change of $y$.
- $y''$ tells us about the rate of change of $y'$.
- $y^{(3)}$ tells us about the rate of change of $y''$.
- and so on...

- We can get another type of approximation of the solution of a DE by using the information encoded in the derivative and the higher derivatives.
Taylor polynomials

- You have a function \( y = f(x) \) which is “difficult” to evaluate.
- Say you already know the function and its first few derivatives at a point \( x = c \).
- We can use this information about \( f \) to evaluate it at some “nearby” point \( (x = x_0) \).
The tangent line — a linear approximation

- We will start by considering the simplest polynomial — a line.
- The tangent line to a curve gives a pretty good approximation over small distances.
- The equation of the tangent line to the curve at $c$ is

$$y - f(c) = f'(c)(x - c) \quad \text{or} \quad y = f(c) + f'(c)(x - c)$$
An example

- Find the tangent line to $y = (x + 1)e^{-x}$ at $x = 2$.

- At $x = 2$ $y = 3e^{-2}$ and $y' = -2e^2$.

- So the tangent line is $(y - 3e^{-2}) = -2e^{-2}(x - 2)$.

- Or approximately $y = 0.4060 - 0.2707(x - 2)$.

- So at $x = 2.5$ the tangent line gives $y = 0.2707$.

- The real function gives $y = 3.5e^{-2.5} = 0.2873$ — so not too bad.
Towards a better approximation

- The tangent line equation has nice properties
  \[ y(x) = f(c) + f'(c)(x - c) \]
  so that
  \[ y(c) = f(c) \]
  \[ y'(c) = f'(c) \]

- Hence our degree 1 polynomial satisfies
  \[ P_1(c) = f(c) \]
  \[ P_1'(c) = f'(c) \]

- The zeroth and first derivatives agree!
**Quadratic approximation**

- $P_2(x)$ — a parabola that approximates $f(x)$ near (about) $x = c$.

\[
\begin{align*}
P_2(x) &= a_0 + a_1(x - c) + a_2(x - c)^2 \\
P_2'(x) &= a_1 + 2a_2(x - c) \\
P_2''(x) &= 2a_2
\end{align*}
\]

- Note that all higher derivatives of $P_2(x)$ are zero.
Quadratic approximation

- From these equations we see that

\[ P_2(c) = a_0 \]
\[ P'_2(c) = a_1 \]
\[ P''_2(c) = 2a_2 \]

- Since we want the parabola to be a good approximation to \( f(x) \) near \( x = c \) we equate

\[ P_2(c) = f(c) \]
\[ P'_2(c) = f'(c) \]
\[ P''_2(c) = f''(c) \]

- So \( a_0 = f(c) \), \( a_1 = f'(c) \) and \( a_2 = \frac{1}{2} f''(c) \).
Quadratic approximation

- Hence we can write
  \[ P_2(x) = f(c) + f'(c)(x - c) + \frac{1}{2} f''(c)(x - c)^2 \]
- Notice that \( P_2(x) = P_1(x) + \frac{1}{2} f''(c)(x - c)^2 \) — so builds on the linear approximation.

An example

- A quadratic approximation to \( f(x) = (x + 1)e^{-x} \) around \( x = 2 \).
- From before \( P_1(x) = 3e^{-2} - 2e^{-2}(x - 2) \). Also \( f''(2) = e^{-2} \).
- So \( P_1(x) = 3e^{-2} - 2e^{-2}(x - 2) + \frac{1}{2} e^{-2}(x - 2)^2 \)
- Gives \( P_2(2.5) = 0.2876 \) compared with \( f(2.5) = 0.2873 \).
General Taylor polynomial of degree $N$

- We find the polynomial
  \[ P_N(x) = a_0 + a_1(x - c) + \cdots + a_N(x - c)^N \]
  so that
  \[
  \begin{align*}
  P_N(c) &= f(c) \\
  P'_N(c) &= f'(c) \\
  P''_N(c) &= f''(c) \\
  \vdots \\
  P^{(N)}_N(c) &= f^{(N)}(c)
  \end{align*}
  \]

- Exercise: Show that $P^{(k)}_N(c) = k! a_k$.
- Note that $P^{(k)}_N(c) = 0$ for $k > N$. 
General Taylor polynomial of degree $N$

The $N^{th}$ degree Taylor polynomial is

$$P_N(x) = f(c) + f'(c)(x - c) + \frac{1}{2!}f''(c)(x - c)^2 + \frac{1}{3!}f^{(3)}(c)(x - c)^3 + \cdots + \frac{1}{N!}f^{(N)}(c)(x - c)^N$$

This approximates $f(x)$ near $x = c$.

Note:

- $P_N(x) = P_{N-1}(x) + \frac{1}{N!}f^{(N)}(c)(x - c)^N$.
- Can replace $f(c)$ by $y(c)$ etc, if we regard $y = y(x)$ rather than $y = f(x)$. 


An example

- Find a quadratic example for \( \sin x \) near \( x = \frac{\pi}{6} \).

- \( P_2(x) = f(c) + f'(c)(x - c) + \frac{1}{2!} f''(c)(x - c)^2 \).

- \( c = \frac{\pi}{6} \) and
  \( f(x) = \sin x, \ f'(x) = \cos x \) and \( f''(x) = -\sin x \).

- So \( f(\frac{\pi}{6}) = \frac{1}{2}, \ f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \) and \( f''(\frac{\pi}{6}) = -\frac{1}{2} \).

- Hence close to \( x = \frac{\pi}{6} \)
  \( \sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{4}(x - \frac{\pi}{6})^2 \)

- Aside:
  \[
  \sin 31^\circ = \sin(30^\circ + 1^\circ) = \sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right)
  \approx \frac{1}{2} + \frac{\sqrt{3}}{2} \frac{\pi}{180} - \frac{1}{4} \left(\frac{\pi}{180}\right)^2
  \]
Another example

- Find \( P_1(x), P_3(x), P_5(x) \) for \( \sin x \) about \( x = 0 \).

\[
\begin{align*}
y(x) &= \sin x & \text{so } y(0) &= 0 \\
y'(x) &= \cos x & \text{so } y'(0) &= 1 \\
y''(x) &= -\sin x & \text{so } y''(0) &= 0 \\
y^{(3)}(x) &= -\cos x & \text{so } y^{(3)}(0) &= -1 \\
y^{(4)}(x) &= \sin x & \text{so } y^{(4)}(0) &= 0 \\
y^{(5)}(x) &= \cos x & \text{so } y^{(5)}(0) &= 1
\end{align*}
\]

- So \( P_5(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \ldots + \frac{1}{5!}y^{(5)}(0)x^5 \)
- Substituting in the above \( P_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \).
- And \( P_3(x) = x - \frac{1}{3!}x^3, P_1(x) = x. \)
A picture of what is going on

- $P_1(x)$
- $P_3(x)$
- $P_5(x)$
- $\sin(x)$

Taylor polynomials
- Linear
- Quadratic
- General
- Example
Taylor’s Theorem

- Not part of this course . . .
- Tells us that for $x = x_0$ near $x = c$ that $f(x_0) - P_N(x_0) \to 0$ as $N \to \infty$.
- There are some conditions on $f(x)$ — it cannot be too nasty.
- Also tells us about finite $N$ — an error formula.
- The region of “small error” gets larger as $N$ increases.
- This all leads to Taylor series — infinite degree Taylor polynomials.
- Warning — domain might be different.
Taylor Series

- For $x$ in some region about $x = c$.

\[
y = f(x) = f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2 + \ldots
\]

- Note — this is not an approximation.
  It is an exact representation of $f(x)$.

- The general term in the series looks like
  \[\frac{1}{n!}f^{(n)}(c)(x - c)^n\]

- So the series is
  \[
f(x) = f(c) + \sum_{n=1}^{\infty} \frac{1}{n!}f^{(n)}(c)(x - c)^n
\]
An example

Calculate the full Taylor series for $\log_e(x + 5)$ about $x = -2$.

- $f(x) = \log_e(x + 5)$ and $c = -2$. Need $f^{(n)}(-2)$.

\[
\begin{align*}
  f(x) &= \log_e(x + 5) & \text{so } f(-2) &= \log_e 3 \\
  f'(x) &= \frac{1}{x+5} & \text{so } f'(-2) &= \frac{1}{3} \\
  f''(x) &= \frac{-1}{(x+5)^2} & \text{so } f''(-2) &= \frac{-1}{3^2} \\
  f^{(3)}(x) &= \frac{2}{(x+5)^3} & \text{so } f^{(3)}(-2) &= \frac{2}{3^3} \\
  f^{(4)}(x) &= \frac{-2 \times 3}{(x+5)^4} & \text{so } f^{(4)}(-2) &= \frac{-2 \times 3}{3^4} \\
  f^{(5)}(x) &= \frac{2 \times 3 \times 4}{(x+5)^5} & \text{so } f^{(5)}(-2) &= \frac{2 \times 3 \times 4}{3^5}
\end{align*}
\]

- A pattern!
An example

- The sign alternates
  — “+” if $n$ is odd and “−” if $n$ is even.
- Numerator is $1 \times 2 \times 3 \times \cdots \times (n - 1)$.
- Denominator is $3^n$.
- So $f^{(n)}(-2) = (-1)^{n+1} \frac{(n-1)!}{3^n}$.

- Now $f(x) = f(-2) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(-2)(x + 2)^n$
- So we have

$$f(x) = \log_e 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{x + 2}{3} \right)^n$$
Some standard series

Taylor series about $x = 0$.

- $e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots = \sum_{n \geq 0} \frac{x^n}{n!}.$
- $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \frac{1}{9!} x^9 - \frac{1}{11!} x^{11} \cdots$
- $\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \frac{1}{8!} x^8 - \frac{1}{10!} x^{10} \cdots$
- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 \cdots = \sum_{n \geq 0} (-1)^n x^n$ only holds when $|x| < 1$.
- $\log_e (1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 \cdots = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$ only holds when $|x| < 1$. 
Another example

Find the Taylor series about $x = 3$ for $y = x^2 - x - 2$.

- Note that if about $x = 0$ then
  
  \[ P_N(x) = P_2(x) = -2 - x + x^2 = f(x) \text{ for } N \geq 2. \]

- For $c = 3$ we have
  
  \[
  \begin{align*}
  y &= x^2 - x - 2 & \text{so } y(3) &= 4 \\
  y' &= 2x - 1 & \text{so } y'(3) &= 5 \\
  y'' &= 2 & \text{so } y''(3) &= 2 \\
  y^{(n)}(x) &= 0 & \text{so } y^{(n)}(3) &= 0 & n > 2
  \end{align*}
  \]

- Hence $y(x) = y(3) + y'(3)(x - 3) + \frac{1}{2!}y''(3)(x - 3)^2$
  
  or
  
  \[ y(x) = 4 + 5(x - 3) + (x - 3)^2. \]
Another example

- So the Taylor series for a parabola ends at the quadratic term no matter which \( c \) we choose.
- More generally the Taylor series of a polynomial of degree \( N \) will end with the term \((x - c)^N\).
- Check

\[
y(x) = 4 + 5(x - 3) + (x - 3)^2 \\
= 4 + 5x - 15 + x^2 - 6x + 9 \\
= x^2 - x - 1 \quad \checkmark
\]

- Only works for polynomials
  For most functions you cannot go from a Taylor polynomial about \( x = c \) to another about \( x = b \).
Series solutions and polynomial approximate solutions to DE

We are given an initial value problem \[
\begin{align*}
\frac{dy}{dx} &= f(x, y) \\
y(c) &= y_0
\end{align*}
\]

- So we can find \(y'(c) = f(c, y_0)\).
- We can then differentiate the DE to get \(y''(x)\) in terms of \(x, y(x)\) and \(y'(x)\).
- Substitute \(x = c\) and get \(y''(c)\).
- Can repeat to get higher derivatives.
- We can use these to get the coefficients of a Taylor series.
- Euler method was “numeric”, this is “symbolic”.
An example

Find a series solution for the IVP

\[ \frac{dy}{dx} = x - y \quad \text{with} \quad y(0) = 2 \]

So \( c = 0 \) and \( y(c) = 2 \) — find Taylor series.

We have

\[
\begin{align*}
y' &= x - y \quad \text{so} \quad y'(0) = -2 \\
y'' &= 1 - y' \quad \text{so} \quad y''(0) = 1 - (-2) = 3 \\
y^{(3)} &= -y'' \quad \text{so} \quad y^{(3)}(0) = -3 \\
y^{(4)} &= -y^{(3)} \quad \text{so} \quad y^{(4)}(0) = 3 \\
y^{(n)} &= -y^{(n-1)} \quad \text{so} \quad y^{(n)}(0) = (-1)^n 3
\end{align*}
\]

So the Taylor series is

\[ y = 2 - 2x + 3\frac{x^2}{2!} - 3\frac{x^3}{3!} + 3\frac{x^4}{4!} - 3\frac{x^5}{5!} + \ldots \]
Example continued

- So the Taylor series is
  \[ y = 2 - 2x + 3 \left( \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots \right) \]

- Since \( e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \), there is nearly \( e^{-x} \) in there:
  \[ y = 2 - 2x + 3 \left( e^{-x} + x - 1 \right) \]

- So the solution is
  \[ y = x - 1 + 3e^{-x} \]

- Is actually valid for all \( x \in \mathbb{R} \).
Another example

Find a quartic approximation for the solution to the IVP

\[
\frac{dy}{dx} = 2xy \quad \text{with} \quad y(0) = 1
\]

- So \( c = 0 \) and \( y(c) = 1 \) — find Taylor polynomial
- We have

\[
\begin{align*}
y' &= 2xy \\
y'' &= 2y + 2xy' \\
y^{(3)} &= 4y' + 2xy'' \\
y^{(4)} &= 6y'' + 2xy^{(3)}
\end{align*}
\]

so \( y'(0) = 0 \)

so \( y''(0) = 2 \times 1 + 0 = 2 \)

so \( y^{(3)}(0) = 0 + 0 = 0 \)

so \( y^{(4)}(0) = 12 + 0 = 12 \)
Another example — continued

So our approximation is $y(x) \approx P_4(x)$.

\[
y(x) \approx y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y^{(3)}(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} = 1 + 0 + \frac{2}{2}x^2 + 0 + \frac{12}{24}x^4
\]

so

\[
y(x) \approx 1 + x^2 + \frac{1}{2}x^4 \quad \text{near } x = 0
\]
Yet another example

\[
\frac{dy}{dx} = x^2 + \frac{1}{2}y^2 \quad \text{with} \quad y(2) = -1
\]

Find a quintic approximation about \( x = 2 \) for \( y(x) \) and use it to estimate \( y(2.1) \).

- So \( c = 2 \) and \( y(c) = -1 \) — find Taylor polynomial
- From the DE we have
  - \( y' = x^2 + \frac{1}{2}y^2 \) so \( y'(2) = 2^2 + \frac{1}{2}(-1)^2 = \frac{9}{2} \).
  - \( y'' = 2x + y'y \) so \( y''(2) = 4 + (-1)\frac{9}{2} = -\frac{1}{2} \).
  - \( y^{(3)} = 2 + (y')^2 + yy'' \) so
    \[
y^{(3)}(2) = 2 + \left(\frac{9}{2}\right)^2 + (-1)(-\frac{1}{2}) = \frac{91}{4}
    \]
Yet another example — continued

Keep going

- \( y^{(4)} = 3y'y'' + yy^{(3)} \) so \( y^{(4)}(2) = -\frac{27}{4} - \frac{91}{4} = -\frac{58}{2} \)

- \( y^{(5)} = 3(y'')^2 + 4y'y'' + yy^{(4)} \) so
  \[
y^{(5)}(2) = \frac{3}{4} + \frac{819}{2} + \frac{59}{2} = \frac{1759}{4} \]

And so

\[
y(x) \approx -1 + \frac{9}{2}(x - 2) - \frac{1}{4}(x - 2)^2 + \frac{91}{24}(x - 2)^3 - \frac{59}{48}(x - 2)^4 + \frac{1759}{480}(x - 2)^5
\]
Yet another example — continued

- So we can now find \( y(2.1) \):

\[
y(2.1) \approx -1 + \frac{9}{2}(0.1) - \frac{1}{4}(0.1)^2 + \frac{91}{24}(0.1)^3 \\
- \frac{59}{48}(0.1)^4 + \frac{1759}{480}(0.1)^5 \\
\approx -0.54880
\]

- Compare this with the Improved Euler method
  - Use 20 steps with \( \Delta x = 0.005 \)
  - Obtain \( y(2.1) \approx -0.54879 \).

- Of course to use Taylor series we need to differentiate (harder for a computer), while for Euler method(s) we just need arithmetic.

- However we can get rigorous error bounds with Taylor series.