Topic 8 — Taylor polynomials and series

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Topic outline

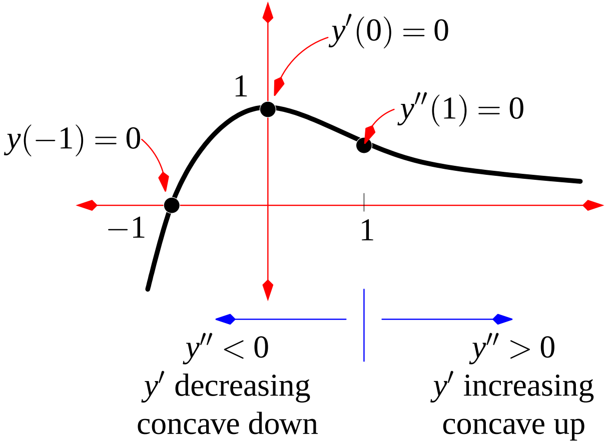
Introduction

- Taylor polynomials — approximation to a function near some point.
- Taylor series — exact representation of a function.

- Not in BZB, but try any standard calculus text such as
  - *Calculus* by Thomas and Finney.
  - *Calculus* by Grossman

Higher derivatives — quick review

- Consider the function $y = (1+x)e^{-x}$.
  Derivatives are $y' = -xe^{-x}$ and $y'' = (x-1)e^{-x}$.



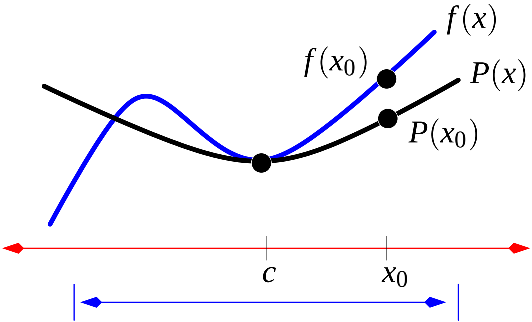
Higher derivatives — quick review

- y' tells us about the rate of change of y.
- y'' tells us about the rate of change of y'.
- $y^{(3)}$ tells us about the rate of change of y''.
- and so on. . .

- We can get another type of approximation of the solution of a DE by using the information encoded in the derivative and the higher derivatives.

Taylor polynomials

- You have a function $y = f(x)$ which is "difficult" to evaluate.
- Say you already know the function and its first few derivatives at a point $x = c$.
- We can use this information about f to evaluate it at some "nearby" point ($x = x_0$).



The tangent line — a linear approximation

- We will start by considering the simplest polynomial — a line.
- The tangent line to a curve gives a pretty good approximation over small distances.
- The equation of the tangent line to the curve at c is

$$\begin{aligned} y - f(c) &= f'(c)(x-c) \qquad \text{or} \\ y &= f(c) + f'(c)(x-c) \end{aligned}$$

An example

- Find the tangent line to $y = (x+1)e^{-x}$ at $x = 2$.

- At $x = 2$ $y = 3e^{-2}$ and $y' = -2e^{2}$.
- So the tangent line is $(y - 3e^{-2}) = -2e^{-2}(x - 2)$.
- Or approximately $y = 0.4060 - 0.2707(x - 2)$.
- So at $x = 2.5$ the tangent line gives $y = 0.2707$.

- The real function gives $y = 3.5e^{-2.5} = 0.2873$ — so not too bad.

Towards a better approximation

- The tangent line equation has nice properties

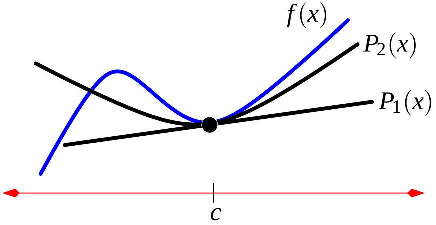
$$\begin{aligned} y(x) &= f(c) + f'(c)(x-c) \\ y(c) &= f(c) \\ y'(c) &= f'(c) \end{aligned} \qquad \text{so that}$$

- Hence our degree 1 polynomial satisfies

$$\begin{aligned} P_1(c) &= f(c) \\ P_1'(c) &= f'(c) \end{aligned}$$

- The zeroth and first derivatives agree!

Quadratic approximation



- $P_2(x)$ — a parabola that approximates $f(x)$ near (about) $x = c$.

$$\begin{aligned} P_2(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 \\ P_2'(x) &= a_1 + 2a_2(x-c) \\ P_2''(x) &= 2a_2 \end{aligned}$$

- Note that all higher derivatives of $P_2(x)$ are zero.

Quadratic approximation

- From these equations we see that

$$\begin{aligned} P_2(c) &= a_0 \\ P_2'(c) &= a_1 \\ P_2''(c) &= 2a_2 \end{aligned}$$

- Since we want the parabola to be a good approximation to $f(x)$ near $x = c$ we equate

$$\begin{aligned} P_2(c) &= f(c) \\ P_2'(c) &= f'(c) \\ P_2''(c) &= f''(c) \end{aligned}$$

- So $a_0 = f(c)$, $a_1 = f'(c)$ and $a_2 = \frac{1}{2}f''(c)$.

Quadratic approximation

- Hence we can write
  $P_2(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2$
- Notice that $P_2(x) = P_1(x) + \frac{1}{2}f''(c)(x-c)^2$ — so builds on the linear approximation.

An example

- A quadratic approximation to $f(x) = (x+1)e^{-x}$ around $x = 2$.
- From before $P_1(x) = 3e^{-2} - 2e^{-2}(x-2)$. Also $f''(2) = e^{-2}$.
- So $P_1(x) = 3e^{-2} - 2e^{-2}(x-2) + \frac{1}{2}e^{-2}(x-2)^2$
- Gives $P_2(2.5) = 0.2876$ compared with $f(2.5) = 0.2873$.

General Taylor polynomial of degree N

- We find the polynomial $P_N(x) = a_0 + a_1(x-c) + \cdots + a_N(x-c)^N$ so that

$$\begin{aligned} P_N(c) &= f(c) \\ P_N'(c) &= f'(c) \\ P_N''(c) &= f''(c) \\ &\vdots \\ P_N^{(N)}(c) &= f^{(N)}(c) \end{aligned}$$

- Exercise: Show that $P_N^{(k)}(c) = k!a_k$.
- Note that $P_N^{(k)}(c) = 0$ for $k > N$.

General Taylor polynomial of degree N

- The N^{th} degree Taylor polynomial is

$$P_N(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \frac{1}{3!}f^{(3)}(c)(x-c)^3 + \cdots + \frac{1}{N!}f^{(N)}(c)(x-c)^N$$

- This approximates $f(x)$ near $x = c$.
- Note:
  - $P_N(x) = P_{N-1}(x) + \frac{1}{N!}f^{(N)}(c)(x-c)^N$.
  - Can replace $f(c)$ by $y(c)$ etc, if we regard $y = y(x)$ rather than $y = f(x)$.

An example

- Find a quadratic example for $\sin x$ near $x = \frac{\pi}{6}$.
- $P_2(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2$.
- $c = \frac{\pi}{6}$ and
  $f(x) = \sin x,\ f'(x) = \cos x$ and $f''(x) = -\sin x$.
- So $f(\frac{\pi}{6}) = \frac{1}{2},\ f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $f''(\frac{\pi}{6}) = -\frac{1}{2}$.
- Hence close to $x = \frac{\pi}{6}$
  $\sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{4}(x - \frac{\pi}{6})^2$
- Aside:

$$\begin{aligned} \sin 31^\circ &= \sin(30^\circ + 1^\circ) = \sin(\frac{\pi}{6} + \frac{\pi}{180}) \\ &\approx \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\pi}{180} - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 \end{aligned}$$

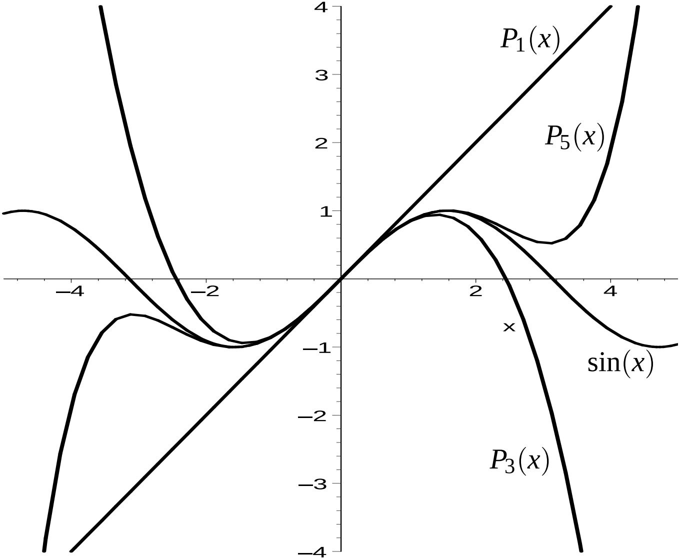
Another example

- Find $P_1(x)$, $P_3(x)$, $P_5(x)$ for $\sin x$ about $x = 0$.

$$\begin{aligned} y &= \sin x & \text{so } y(0) &= 0 \\ y' &= \cos x & \text{so } y'(0) &= 1 \\ y'' &= -\sin x & \text{so } y''(0) &= 0 \\ y^{(3)} &= -\cos x & \text{so } y^{(3)}(0) &= -1 \\ y^{(4)} &= \sin x & \text{so } y^{(4)}(0) &= 0 \\ y^{(5)} &= \cos x & \text{so } y^{(5)}(0) &= 1 \end{aligned}$$

- So $P_5(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \ldots \frac{1}{5!}y^{(5)}(0)x^5$
- Substituting in the above $P_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$.
- And $P_3(x) = x - \frac{1}{3!}x^3$, $P_1(x) = x$.

A picture of what is going on



Taylor's Theorem

- Not part of this course . . .
- Tells us that for $x = x_0$ near $x = c$ that $f(x_0) - P_N(x_0) \to 0$ as $N \to \infty$.
- There are some conditions on $f(x)$ — it cannot be too nasty.
- Also tells us about finite N — an error formula.
- The region of "small error" gets larger as N increases.
- This all leads to Taylor series — infinite degree Taylor polynomials.
- Warning — domain might be different.

Taylor Series

- For x in some region about $x = c$.

$$y = f(x) = f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2 + \dots$$

- Note — this is not an approximation.
  It is an exact representation of $f(x)$.
- The general term in the series looks like
  $\frac{1}{n!}f^{(n)}(c)(x - c)^n$
- So the series is

$$f(x) = f(c) + \sum_{n=1}^{\infty} \frac{1}{n!}f^{(n)}(c)(x - c)^n$$

An example

Calculate the full Taylor series for $\log_e(x+5)$ about $x = -2$.

- $f(x) = \log_e(x+5)$ and $c = -2$. Need $f^{(n)}(-2)$.

$$\begin{aligned}
f(x) &= \log_e(x+5) & &\text{so } f(-2) = \log_e 3 \\
f'(x) &= \tfrac{1}{x+5} & &\text{so } f'(-2) = \tfrac{1}{3} \\
f''(x) &= \tfrac{-1}{(x+5)^2} & &\text{so } f''(-2) = \tfrac{-1}{3^2} \\
f^{(3)}(x) &= \tfrac{+2}{(x+5)^3} & &\text{so } f^{(3)}(-2) = \tfrac{2}{3^3} \\
f^{(4)}(x) &= \tfrac{-2\times 3}{(x+5)^4} & &\text{so } f^{(4)}(-2) = \tfrac{-2\times 3}{3^4} \\
f^{(5)}(x) &= \tfrac{2\times 3\times 4}{(x+5)^5} & &\text{so } f^{(5)}(-2) = \tfrac{2\times 3\times 4}{3^5}
\end{aligned}$$

- A pattern!

An example

- The sign alternates
  — "$+$" if n is odd and "$-$" if n is even.
- Numerator is $1 \times 2 \times 3 \times \cdots \times (n-1)$.
- Denominator is 3^n.
- So $f^{(n)}(-2) = (-1)^{n+1}\frac{(n-1)!}{3^n}$.
- Now $f(x) = f(-2) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(-2)(x+2)^n$
- So we have

$$f(x) = \log_e 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(\frac{x+2}{3}\right)^n$$

Some standard series

Taylor series about $x = 0$.

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n\geq 0} \frac{x^n}{n!}$.
- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} \dots$
- $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} \dots$
- $(1+x)^{-1} = 1-x+x^2-x^3+x^4-x^5 \cdots = \sum_{n\geq 0}(-1)^n x^n$ only holds when $|x| < 1$.
- $\log_e(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4 \cdots = \sum_{n\geq 1}(-1)^{n+1}\frac{x^n}{n}$ only holds when $|x| < 1$.

Another example

Find the Taylor series about $x = 3$ for $y = x^2 - x - 2$.

- Note that if about $x = 0$ then $P_N(x) = P_2(x) = -2 - x + x^2 = f(x)$ for $N \geq 2$.
- For $c = 3$ we have

$$\begin{aligned} y &= x^2 - x - 2 & \text{so } y(3) &= 4 \\ y' &= 2x - 1 & \text{so } y'(3) &= 5 \\ y'' &= 2 & \text{so } y''(3) &= 2 \\ y^{(n)}(x) &= 0 & \text{so } y^{(n)}(3) &= 0 \qquad n > 2 \end{aligned}$$

- Hence $y(x) = y(3) + y'(3)(x-3) + \frac{1}{2!}y''(3)(x-3)^2$ or $y(x) = 4 + 5(x-3) + (x-3)^2$.

Another example

- So the Taylor series for a parabola ends at the quadratic term no matter which c we choose.
- More generally the Taylor series of a polynomial of degree N will end with the term $(x-c)^N$.
- Check

$$\begin{aligned} y(x) &= 4+5(x-3)+(x-3)^2 \\ &= 4+5x-15+x^2-6x+9 \\ &= x^2-x-1 \qquad \checkmark \end{aligned}$$

- Only works for polynomials
  For most functions you cannot go from a Taylor polynomial about $x=c$ to another about $x=b$.

Series solutions and polynomial approximate solutions to DE

We are given an initial value problem $\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} & = f(x,y) \\ y(c) & = y_0 \end{cases}$

- So we can find $y'(c) = f(c, y_0)$.
- We can then differentiate the DE to get $y''(x)$ in terms of x, $y(x)$ and $y'(x)$.
- Substitute $x = c$ and get $y''(c)$.
- Can repeat to get higher derivatives.
- We can use these to get the coefficients of a Taylor series.
- Euler method was "numeric", this is "symbolic".

An example

Find a series solution for the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y \qquad \text{with} \qquad y(0) = 2$$

- So $c = 0$ and $y(c) = 2$ — find Taylor series.
- We have

$$\begin{array}{lll} y' = x - y & \text{so} & y'(0) = -2 \\ y'' = 1 - y' & \text{so} & y''(0) = 1 - (-2) = 3 \\ y^{(3)} = -y'' & \text{so} & y^{(3)}(0) = -3 \\ y^{(4)} = -y^{(3)} & \text{so} & y^{(4)}(0) = 3 \\ y^{(n)} = -y^{(n-1)} & \text{so} & y^{(n)}(0) = (-1)^n 3 \end{array}$$

- So the Taylor series is
  $y = 2 - 2x + 3\frac{x^2}{2!} - 3\frac{x^3}{3!} + 3\frac{x^4}{4!} - 3\frac{x^5}{5!} + \ldots$

Example continued

- So the Taylor series is
  $y = 2 - 2x + 3\left(\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)$
- Since $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$, there is nearly e^{-x} in there:
  $y = 2 - 2x + 3\left(e^{-x} + x - 1\right)$
- So the solution is

$$y = x - 1 + 3e^{-x}$$

- Is actually valid for all $x \in \mathbb{R}$.

Another example

Find a quartic approximation for the solution to the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy \qquad \text{with} \qquad y(0) = 1$$

- So $c = 0$ and $y(c) = 1$ — find Taylor polynomial
- We have

$$\begin{aligned}
y' &= 2xy & \text{so} \quad & y'(0) = 0 \\
y'' &= 2y + 2xy' & \text{so} \quad & y''(0) = 2 \times 1 + 0 = 2 \\
y^{(3)} &= 4y' + 2xy'' & \text{so} \quad & y^{(3)}(0) = 0 + 0 = 0 \\
y^{(4)} &= 6y'' + 2xy^{(3)} & \text{so} \quad & y^{(4)}(0) = 12 + 0 = 12
\end{aligned}$$

Another example — continued

- So our approximation is $y(x) \approx P_4(x)$.

$$
\begin{aligned}
y(x) &\approx y(0) + y'(0)x + y''(0)\frac{x^2}{2!} \\
&\qquad + y^{(3)}(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} \\
&= 1 + 0 + \frac{2}{2}x^2 + 0 + \frac{12}{24}x^4
\end{aligned}
$$

so

$$
y(x) \approx 1 + x^2 + \frac{1}{2}x^4 \qquad \text{near } x = 0
$$

Yet another example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{1}{2}y^2 \qquad \text{with} \qquad y(2) = -1$$

Find a quintic approximation about $x = 2$ for $y(x)$ and use it to estimate $y(2.1)$.

- So $c = 2$ and $y(c) = -1$ — find Taylor polynomial
- From the DE we have
  - $y' = x^2 + \frac{1}{2}y^2$ so $y'(2) = 2^2 + \frac{1}{2}(-1)^2 = \frac{9}{2}$.
  - $y'' = 2x + y'y$ so $y''(2) = 4 + (-1)\frac{9}{2} = -\frac{1}{2}$.
  - $y^{(3)} = 2 + (y')^2 + yy''$ so $y^{(3)}(2) = 2 + (\frac{9}{2})^2 + (-1)(-\frac{1}{2}) = \frac{91}{4}$

Yet another example — continued

- Keep going
  - $y^{(4)} = 3y'y'' + yy^{(3)}$ so $y^{(4)}(2) = -\frac{27}{4} - \frac{91}{4} = -\frac{58}{2}$
  - $y^{(5)} = 3(y'')^2 + 4y'y'' + yy^{(4)}$ so $y^{(5)}(2) = \frac{3}{4} + \frac{819}{2} + \frac{59}{2} = \frac{1759}{4}$
- And so

$$y(x) \approx -1 + \frac{9}{2}(x-2) - \frac{1}{4}(x-2)^2 + \frac{91}{24}(x-2)^3 - \frac{59}{48}(x-2)^4 + \frac{1759}{480}(x-2)^5$$

Yet another example — continued

- So we can now find $y(2.1)$:

$$
\begin{aligned}
y(2.1) \approx -1 + \frac{9}{2}(0.1) - \frac{1}{4}(0.1)^2 + \frac{91}{24}(0.1)^3 \\
- \frac{59}{48}(0.1)^4 + \frac{1759}{480}(0.1)^5 \\
\approx -0.54880
\end{aligned}
$$

- Compare this with the Improved Euler method
  - Use 20 steps with $\Delta x = 0.005$
  - Obtain $y(2.1) \approx -0.54879$.
- Of course to use Taylor series we need to differentiate (harder for a computer), while for Euler method(s) we just need arithmetic.
- However we can get rigorous error bounds with Taylor series.