3.1 (a) Events $A$ and $B$ are such that $\Pr(A) = 0.2$, $\Pr(B) = \beta$ and $\Pr(A \cup B) = 0.6$.

Find $\beta$ if

i. $A$ and $B$ are mutually exclusive;

ii. $A$ and $B$ are independent;

and show that $0.4 \leq \beta \leq 0.6$.

(b) Drug $A$ causes an allergic reaction in $3\%$ of adults, drug $B$ in $6\%$, while $0.4\%$ are allergic to both. What sort of relationship exists between allergic reactions to the drugs $A$ and $B$?

(c) If $\Pr(A) = 0.4$, $\Pr(B \mid A) = 0.1$ and $\Pr(B \mid A') = 0.6$, find $\Pr(B)$ and $\Pr(A \mid B)$.

3.2 (a) A Swedish study (see http://www.bbc.co.uk/2/hi/health/4051331.stm) followed 61,084 women over 13 years. Suppose 9,877 women consumed more than four servings of dairy products per day. Of these women, 28 were diagnosed with ovarian cancer. Of the 29,452 women who consumed less than two servings of dairy products per day, 44 were diagnosed with ovarian cancer. Give an estimate of the risk of developing ovarian cancer for women consuming more than four servings of dairy products per day, relative to those women who consume less than two servings.

(b) The Chinese Mini-Mental Status Test (CMMS) is a test consisting of 114 items intended to identify people with Alzheimer’s disease and senile dementia among people in China. Low test scores are taken to indicate the presence of dementia. An extensive clinical evaluation was performed of this instrument, whereby participants were interviewed by experts and definitive diagnosis of dementia was made.

Following shows the results obtained on a subgroup of people.

<table>
<thead>
<tr>
<th>CMMS score</th>
<th>Nondemented</th>
<th>Demented</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–5</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6–10</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>11–15</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>16–20</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>21–25</td>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>26–30</td>
<td>18</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>46</td>
<td>16</td>
</tr>
</tbody>
</table>

Suppose a score of $\leq 20$ on the test is used to identify people with dementia. Assume that the data above are representative of the underlying probabilities.

i. What is the sensitivity of the test?

ii. What is the specificity of the test?

iii. If $1\%$ of a community has dementia, what is the positive predictive value of the test?

iv. How would these values change if the threshold score was changed to 15? Comment.
Tutorial problems

3.3 The level of prostate-specific antigen (PSA) in the blood is frequently used as a screening test for prostate cancer. A report gives the following data regarding the relationship between a positive PSA test ($\geq 5$ ng/dL) and prostate cancer.

<table>
<thead>
<tr>
<th>PSA test result</th>
<th>Prostate cancer</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>92</td>
</tr>
<tr>
<td>+</td>
<td>−</td>
<td>27</td>
</tr>
<tr>
<td>−</td>
<td>+</td>
<td>46</td>
</tr>
<tr>
<td>−</td>
<td>−</td>
<td>568</td>
</tr>
</tbody>
</table>

i. Use these data to estimate the sensitivity, specificity and positive predictive value of the test?

ii. How might these data have been obtained?

3.4 Suppose that among males aged 50–59 the PSA level is given by the following graphs, according to whether the individual has prostate cancer or does not. These graphs give the cumulative probability $F(x) = \Pr(PSA \leq x)$. This is called the cumulative distribution function and is equivalent to a population cumulative relative frequency.

Suppose we choose to say the PSA test is “positive”, if the PSA level is greater than $\ell$, i.e. $P = \{PSA > \ell\}$.

Assume that the prevalence of prostate cancer in this age group is 20%.

Find the sensitivity, specificity, positive predictive value, percentage false-positive and percentage false-negative for $\ell = 4, 5, 6, 7$.

Discuss the effects of these different levels. How would you choose what is “best”?

The ROC curve plots sensitivity against specificity. Sketch the ROC curve. Does this help in choosing a “best” test?

3.5 Sam has an alarm which rings with probability 0.7. If it rings, it will wake him with probability 0.8 and then he will be in time for his 9am lecture. If it doesn’t ring he will wake up in time for his 9am lecture with probability 0.4. What is the probability that he makes it in time for the 9am lecture?

3.6 There are three coins in a box. One is a two-headed coin, another is a two-tailed coin, and the third is a fair coin. When one of the coins is chosen at random and tossed, it shows heads. What is the probability that it is the fair coin?
Review problems

3.7 (a) What is the frequentist definition of probability?
(b) Explain the difference between mutually exclusive and independent events.
(c) Give an example of positively related events; of negatively related events.
(d) What is the value of Bayes’ theorem? How is it applied in diagnostic testing?
   (Who was Bayes anyway?)
(e) What is relative risk?
   (Why can’t we estimate it from a case-control study? What else do we need?)
(f) What is sensitivity? What is specificity?
(g) What is a false negative? What is a false positive? Why are these bad?
(h) What is positive predictive value? What is negative predictive value?

3.8 State whether each of the following statements is true for all events $A$ and $B$, or false (i.e., not true for all events $A$ and $B$):

(a) $\Pr(A) = 1 - \Pr(A')$;
(b) $\Pr(A \cap B) = \Pr(A) \Pr(B)$;
(c) $\Pr(A) \leq \Pr(A \cup B)$;
(d) $\Pr(A \cup B) = \Pr(A) + \Pr(B)$;
(e) $\Pr(A \cap B) = 1 - \Pr(A' \cap B')$
(f) $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$;
(g) $\Pr(A \cap B) = \Pr(A) - \Pr(A \cap B')$;
(h) $\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$;
(i) $\Pr(A \cap B) \leq \Pr(A) \Pr(B)$;
(j) $\Pr(A) \Pr(A') \leq \frac{1}{4}$.

3.9 State whether the each of following is true or false — prove or explain your answer.

(a) If $B \subseteq A$ then $\Pr(A \mid B) = 1$;
(b) $\Pr(A' \mid B) = 1 - \Pr(A \mid B)$;
(c) If $A$ and $B$ are mutually exclusive, then $\Pr(A \mid B) = \Pr(B)$;
(d) $\Pr(A' \mid B) = \Pr(A \mid B')$;
(e) $\Pr(A \mid B) + \Pr(A \mid B') = 1$. 
Additional problems

3.10 A large number of drivers were questioned and classified according to age and number of accidents in the last year. The relative frequencies are given below:

<table>
<thead>
<tr>
<th>Age</th>
<th>More than one accident</th>
<th>At most one accident</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 30</td>
<td>0.47</td>
<td>0.13</td>
</tr>
<tr>
<td>⩾ 30</td>
<td>0.23</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Show that age and number of accidents are not independent. Does this information imply that young drivers are worse drivers? Explain.

3.11 Consider an alarm system as indicated in the diagram, for which each of the switches independently works in case of emergency with probability 0.9. What is the reliability of the system? i.e., the probability that at least one of the switches works in case of emergency.

3.12 Let $A$ and $B$ be events in an outcome space $\Omega$ such that $\Pr(A) = \alpha$, $\Pr(B) = \beta$ and $\Pr(A \cap B) = \gamma$. Find expressions for the probabilities of each of the following events in terms of $\alpha$, $\beta$, $\gamma$: $A \cup B$; $A' \cap B$; $A' \cap B'$; $A' \cup B'$.

3.13 A test for detecting a certain type of cancer gives a positive result for 95% of a large number of patients subsequently to found to have the cancer, and gave a negative result for 99% of those not having the cancer. If the test is applied randomly to a population in which the proportion of persons having the cancer is 1%, find the probability that a person has the cancer if his test gave a positive result.

3.14 $A$ and $B$ are events with $\Pr(A) = 0.3$, $\Pr(B) = 0.8$ and $\Pr(A \cap B) = 0.2$.

(a) Find $\Pr(A')$, $\Pr(A \cup B)$, $\Pr(A \cap B')$ and $\Pr(A' \cup B)$.

(b) Are $A$ and $B$ mutually exclusive?

(c) Are $A$ and $B$ independent, positively related or negatively related?

3.15 A researcher looked at the relationship between parental smoking and the incidence of pneumonia and/or bronchitis in children in the first year of life. She found that 7.8% of children with nonsmoking parents had episodes of pneumonia and/or bronchitis in the first year of life, whereas, respectively, 11.4% of children with one smoking parent and 17.6% of children with two smoking parents had such an episode. Suppose that in the general population both parents are smokers in 40% of households, one parent smokes in 25% of households, and neither parent smokes in 35% of households. What percentage of children in the general population will have pneumonia and/or bronchitis in the first year of life?

3.16 A community-based study of respiratory illness during the first year of life was conducted. As part of this study, a group of children were classified according to family socioeconomic status. The numbers of children in each group who experienced persistent respiratory symptoms are shown below.

<table>
<thead>
<tr>
<th>Socioeconomic Status</th>
<th>Number of Children</th>
<th>Number with Symptoms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>79</td>
<td>31</td>
</tr>
<tr>
<td>Middle</td>
<td>122</td>
<td>29</td>
</tr>
<tr>
<td>High</td>
<td>192</td>
<td>27</td>
</tr>
</tbody>
</table>

(a) Assuming the numbers are high enough to satisfy the frequentist definition of probability, compute the probability of children suffering persistent respiratory symptoms in each socioeconomic group.

(b) Does there appear to be an association between socioeconomic status and respiratory symptoms?
3.17 You have torn a tendon and are facing surgery to repair it. The orthopedic surgeon explains the risks to you: Infection occurs in 3% of such operations, the repair fails in 14%, and both infection and failure occur together in 1%.

(a) What percentage of these operations succeed and are free from infection?
(b) Are failure and infection independent events? Explain. If they are not, determine if they are positively or negatively correlated.

3.18 The National Institute for Occupational Safety and health has developed a case definition of carpal tunnel syndrome—an affliction of the wrist—that incorporates three criteria: symptoms of nerve involvement, a history of occupational risk factors and the presence of physical exam findings. The sensitivity of this definition as a test for carpal tunnel syndrome is 0.67; its specificity is 0.58.

(a) In a population in which the prevalence of carpal tunnel syndrome is estimated to be 15%, what is the predictive value of a positive test result?
(b) How does this predictive value change if the prevalence is only 10%? If it is 5%? Comment.

3.19 The \( \epsilon^4 \) allele of the gene encoding apolipoprotein E (APOE) is strongly associated with Alzheimer’s disease (AD), but its value in making the diagnosis remains uncertain. A study was conducted among 2188 patients who were evaluated at autopsy for AD by previously established pathological criteria. Patients were also evaluated clinically for the presence of AD. The data are presented below.

<table>
<thead>
<tr>
<th>Pathological diagnosis</th>
<th>Alzheimer’s disease</th>
<th>Other causes of dementia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clinical diagnosis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alzheimer’s disease</td>
<td>1643</td>
<td>190</td>
</tr>
<tr>
<td>Other causes of dementia</td>
<td>127</td>
<td>228</td>
</tr>
</tbody>
</table>

Suppose the pathological diagnosis is considered to be the gold standard for AD.

(a) Consider the clinical diagnosis as a screening test for AD.
   i. What is the sensitivity of this test?
   ii. What is the specificity of this test?
   iii. Suppose this test is applied to the residents of an aged home where the prevalence of AD is 3%. What is the positive predictive value of the test?

(b) To possibly improve on the diagnostic accuracy of the clinical diagnosis for AD, information on both the APOE genotype as well as the clinical diagnosis were considered. The data are shown below:

<table>
<thead>
<tr>
<th>APOE genotype</th>
<th>Both clinical and pathological criteria for AD</th>
<th>Only clinical criteria for AD</th>
<th>Only pathological criteria for AD</th>
<th>Neither clinical nor pathological criteria for AD</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \geq 1 \epsilon^4 ) allele</td>
<td>1076</td>
<td>66</td>
<td>66</td>
<td>67</td>
</tr>
<tr>
<td>No ( \epsilon^4 ) allele</td>
<td>567</td>
<td>124</td>
<td>61</td>
<td>161</td>
</tr>
<tr>
<td>Total</td>
<td>1643</td>
<td>190</td>
<td>127</td>
<td>228</td>
</tr>
</tbody>
</table>

Suppose we consider the combination of both a clinical diagnosis for AD and the presence of \( \geq 1 \epsilon^4 \) allele as a screening test for AD.

i. What is the sensitivity of this test?
ii. What is the specificity of this test?
iii. Is this a better test than the one above?

3.20 The sensitivity of the mammogram, a screening test for detecting breast cancer, is 0.85; its specificity is 0.80. In a population in which the probability that a randomly chosen woman has breast cancer is 0.0025, what is the probability that a woman has breast cancer given than her mammogram is positive?