The University of Melbourne

Semester 1 Exam — June, 2007

Department of Mathematics and Statistics

620-201 Probability

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 5 pages

Authorised materials:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room. Note: Electronic calculators are NOT permitted.

Instructions to Invigilators:

Students may take this exam paper with them at the end of the exam.

Instructions to Students:
This paper has **ten** (10) questions.
Attempt as many questions, or parts of questions, as you can.
Questions carry marks as shown in the brackets after the question statement.
The total of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. For two events A and B it is known that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$ and $P(A \cap B^c) = \frac{1}{9}$.

   (a) Find $P(B^c)$.

   (b) Find $P(A \cup B)$.

   (c) Find $P(A|B)$.

   (d) Find $P(B^c|A)$.

   (e) Are A and B positively or negatively related ? Justify your answer.

   (f) For events C and D we have $P(C) = 0.3, P(D) = \alpha$, and $P(C \cup D) = 0.7$.

      i. Find α if C and D are mutually exclusive;

      ii. Find α if C and D are independent;

      iii. Show that in general $0.4 \leq \alpha \leq 0.7$.

   (g) Consider a random experiment with sample space Ω.

      i. Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

      ii. Using the axioms, prove that for any two events E and F, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

[15 marks]

2. (a) Let A and B be events in a random experiment. State Bayes' Formula for the probability $P(B|A)$.

   (b) A student sits a multiple choice exam that has m choices of answer for each question. Assume that the probability that the student knows the correct answer to a question is p. If the student does not know the correct answer then he marks an answer at random with all choices equally likely to be chosen.

      i. What is the probability that the correct answer is marked for a given question?

      ii. Suppose that the answer marked to a particular question is in fact correct. What is the probability that the student actually knew the answer (rather than guessing it) ?

      iii. If $p = 0.2$ and $m = 4$, find the standard deviation of the number of correct answers obtained by guessing on a 25 question exam (assume the questions are answered independently).

[7 marks]

3. (a) An urn contains 4 defective and 8 non-defective items. Let N be the number of defective items in a sample of size 6 chosen from the urn.

      i. Assume the sample is chosen at random and with replacement. Giving your reasons, name the distribution of the random variable N and give the value of any parameter(s) ? What is the variance of N ?

      ii. Assume the sample is chosen at random and without replacement. Giving your reasons, name the distribution of the random variable N. List the possible values of N and write down a formula for its probability mass function (there is no need to calculate the numerical probabilities).

(b) A telephone exchange opens at 9am and the waiting time T (in minutes) between incoming calls has an exponential distribution ie $T \stackrel{d}{=} \exp(\frac{1}{2})$. Let X be the waiting time between the 2nd and 4th calls received after 9am and let Y be the waiting time till the 70th call.

i. Giving your reasons, name the distribution of the random variable X and give the value of any parameter(s) ? Calculate $P(X \leq 2 \text{ minutes})$ (you may leave your answer as a function of e).

ii. Giving your reasons, name the distribution of the random variable Y and give the value of any parameter(s). Also state an approximate distribution for Y.

(c) Let $U \stackrel{d}{=} R(0,1)$ ie U is uniformly distributed on $(0,1)$, and let $Y = -\frac{1}{5}\ln(1-U)$. Giving your reasons, name the distribution of the random variable Y and give the value of any parameter(s) ? (Note: no calculations are required).

(d) Consider $X \stackrel{d}{=} N(0,1)$ (with probability density function ϕ) and let

$$Y = \int_{-\infty}^{X} \phi(z)dz.$$

Giving your reasons, name the distribution of the random variable Y and give the value of any parameter(s) ? (Note: no calculations are required).

[14 marks]

4. (a) Let X be a continuous random variable with pdf (probability density function)

$$f_X(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = 2X - 1$, what are the possible values of Y ? Find both the distribution function $F_Y(y)$ and pdf $f_Y(y)$ of Y. In your answer be careful to give expressions for all values of $y \in \mathbb{R}$.

(b) Let T be a continuous random variable with pdf (probability density function)

$$f_T(t) = \begin{cases} \frac{1}{2}(t+1) & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of $Z = T^2$ and $E(Z)$.

[14 marks]

5. Let X and Y be two independent random variables. Assume that X has a continuous distribution with pdf (probability density function)

$$f_X(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that Y has a continuous uniform distribution on the interval $(0, 1)$, namely $Y \stackrel{d}{=} R(0, 1)$. Let $Z = X + Y$.

(a) Write down the convolution formula for the probability density function $f_Z(z)$ of Z.

(b) Hence derive $f_Z(z)$, carefully explaining each step of your derivation.

[8 marks]

6. Let X and Y have joint probability density function

$$f_{(X,Y)}(x, y) = \begin{cases} 8xy & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Draw a diagram identifying the region in the xy plane where the joint probability density function is non-zero.

(b) Find the marginal probability density functions (pdf's) of X and Y.

(c) Find the conditional pdf of X given $Y = y$, where $0 < y < 1$.

(d) Find the random variable $E(X|Y)$.

(e) Verify that $E(X) = E(E(X|Y))$.

(f) Giving your reasons, state whether X and Y are dependent or independent.

[10 marks]

7. Consider the random sum

$$T = \sum_{k=1}^{N} X_k$$

where the random variable $N \stackrel{d}{=} Pn(\lambda)$ and, for $k = 1, 2, \ldots$, the X_k are independent random variables with $E(X_k) = \mu$ and $V(X_k) = \sigma^2$. Note also that N is independent of all the X_k.

(a) Showing all steps, find the random variables $E(T|N)$ and $V(T|N)$.

(b) Hence find $E(T)$ and $V(T)$.

(c) Calculate the percentage reduction in the variability of T which would occur if instead of being Poisson the distribution of N was degenerate at λ ie $P(N = \lambda) = 1$.

[6 marks]

8. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, and let $S_n = X_1 + X_2 + \ldots + X_n$. Then the Central Limit Theorem (CLT) states that

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1), \quad \text{as } n \to \infty.$$

Prove the CLT by showing that the moment generating function $M_{Z_n}(t)$ converges to $M_Z(t)$ as $n \to \infty$, where $M_Z(t) = \exp\left(\frac{t^2}{2}\right)$ is the moment generating function of the standard normal distribution.

[6 marks]

9. Consider a random variable $X \stackrel{d}{=} \exp(\alpha)$.

   (a) Showing all your steps, derive $M_X(t)$, the moment generating function of X.

   (b) For what values of t is $M_X(t)$ defined?

   (c) Using an appropriate power series expansion, find a general expression for the kth non-central moment μ_k of X and hence find $E(X)$ and $V(X)$.

   (d) Using only your result in (a) and known properties of moment generating functions, derive the mgf for the random variable $Y \stackrel{d}{=} \gamma(r, \alpha)$ (where r is an integer). Hence find $E(Y)$.

[8 marks]

10. (a) Consider the Branching Process $\{X_n, n = 0, 1, 2, 3, \ldots\}$ where X_n is the population size at the nth generation. Assume $P(X_0 = 1) = 1$ and that the pgf of the common offspring distribution N is

$$A(z) = \frac{1}{c}(4 + 3z + 5z^2 + 2z^3)$$

   i. Find the constant c.

   ii. Find $P(N = 2)$, $E(N)$, and $E(N^2)$.

   iii. Let q be the extinction probability for this branching process. Write down an equation satisfied by q. Then find the value of q.

   (b) Consider two urns A and B which contain a total of 3 balls. Let X_n denote the number of balls in urn A at time n. At each time $n = 1, 2, \ldots$ a ball is selected at random to be moved from its current urn to the other one. At each time all balls have the same chance of being chosen, irrespective of which urn they are in. It follows that $\{X_n : n = 0, 1, 2, \ldots\}$ is a discrete time Markov Chain.

   i. Write down the transition probability matrix P for this Markov chain.

   ii. Assume that the initial state of the chain X_0 is uniformly distributed over all possible states. Find $P(X_2 = 2)$.

   iii. Calculate the equilibrium probabilities for this Markov Chain.

[12 marks]

End of the exam