Solutions to Homework 10

Tutors to mark questions 2 and 3

1. (a) 
\[ \eta(y) = E(X^Y | Y = y) = E(X^y) = \begin{cases} E(X^0) = E(1) = 1, & y = 0, \\ E(X^1) = E(X) = \frac{1}{2}, & y = 1. \end{cases} \]

Thus, \( E(Z|Y) = \eta(Y) = 1 - \frac{Y}{2} \). (Note that the straight line through the points \((0, \eta(0)) = (0, 1)\) and \((1, \eta(1)) = (1, \frac{1}{2})\) is \( y^* = 1 - \frac{x^*}{2} \) - we replace \( x^* \) with \( Y \) and \( y^* \) with \( \eta(Y) \)).

(b) 
\[ \zeta(y) = V(X^Y | Y = y) = V(X^y) = \begin{cases} V(X^0) = V(1) = 0, & y = 0 \\ V(X^1) = V(X) = \frac{1}{12}, & y = 1. \end{cases} \]

Thus, \( V(Z|Y) = \zeta(Y) = \frac{Y}{12} \). (Note that the straight line through the points \((0, \zeta(0)) = (0, 0)\) and \((1, \zeta(1)) = (1, \frac{1}{12})\) is \( y^* = \frac{x^*}{12} \) - we replace \( x^* \) with \( Y \) and \( y^* \) with \( \zeta(Y) \)).

(c) 
\[ E(Z) = E(E(Z|Y)) = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}. \]

(d) 
\[ V(Z) = E(V(Z|Y)) + V(E(Z|Y)) = \left( 0 \times \frac{1}{2} + \frac{1}{12} \times \frac{1}{2} \right) + E(\eta(Y)^2) - E(\eta(Y))^2 \]
\[ = \frac{1}{24} + \left( 1^2 \times \frac{1}{2} + \left( \frac{1}{2} \right)^2 \times \frac{1}{2} \right) - \left( \frac{3}{4} \right)^2 \]
\[ = \frac{1}{24} + \frac{5}{8} - \frac{9}{16} \]
\[ = \frac{5}{48}. \]

2. We have that \( N \overset{d}{=} Pn(\Lambda) \) where \( \Lambda \overset{d}{=} \text{exp}(1) \). Then \( P(N = i) = \left( \frac{1}{2} \right)^{i+1} \implies N \overset{d}{=} G(\frac{1}{2}) \). Therefore, \( V(N) = \frac{1 - p}{p^2} = 2 \). We also have that \( \eta(\lambda) = E(N|\Lambda = \lambda) = \lambda \), therefore \( E(N|\Lambda) = \Lambda \). Also, \( \zeta(\lambda) = V(N|\Lambda = \lambda) = \lambda \), therefore \( V(N|\Lambda) = \Lambda \). Thus, \( V(N) = E(V(N|\Lambda)) + V(E(N|\Lambda)) = E(\Lambda) + V(\Lambda) = 1 + 1 = 2 \).

3. Use the inverse transformation method to get \( X \overset{d}{=} -\frac{1}{\lambda} \log(1 - U) \) where \( U \overset{d}{=} R(0, 1) \). From Part 1. \( E(T) = \frac{10}{\lambda} \) and \( V(T) = \frac{20}{\lambda^2} \). If, for example, \( \lambda = \frac{1}{2} \), simulation should give \( E(T) \approx 20 \) and \( V(T) \approx 80 \).
4. We have that $X \sim R(0, 1)$, $Y = \sqrt{X} = \phi(X)$, and $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$.

(a) $\psi(x) = x^{\frac{1}{2}} \implies \psi(\mu) = \frac{1}{\sqrt{2}}$.

\[\psi'(x) = \frac{1}{2} x^{-\frac{1}{2}} \implies \psi'(\mu) = \frac{1}{\sqrt{2}}.\]

\[\psi''(x) = -\frac{1}{4} x^{-\frac{3}{2}} \implies \psi''(\mu) = -\frac{1}{\sqrt{2}}.\]

\[E(\psi(X)) \approx \frac{1}{\sqrt{2}} + \frac{1}{2} \left(-\frac{1}{\sqrt{2}}\right) \frac{1}{12} = \frac{23}{24\sqrt{2}}.\]

\[V(X) \approx \left(\frac{1}{\sqrt{2}}\right)^2 \frac{1}{12} = \frac{1}{24}.\]

(b) $l(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} (x - \mu)$ and $q(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} (x - \mu) - \frac{1}{2\sqrt{2}} (x - \mu)^2$.

The approximation is good as $l(x)$ and $q(x)$ are close to $\psi(x)$ when $x \in (0, 1)$.

(c) The simulated values for $E(Y)$ and $V(Y)$ are approximately equal to $E(\psi(X))$ and $V(\psi(X))$ given in Part (a) above.