These questions relate to topics covered in the last few lectures and so not included in the tutorial problems. There are also some additional practice questions for probability generating functions (a topic not covered in Ghahramani).

**Probability generating functions**

1. Suppose that $X$ has pmf given by

   $$
   \begin{array}{c|cccc}
   x & 0 & 1 & 2 & 3 \\
   \hline
   p_X(x) & 0.4 & 0.3 & 0.2 & 0.1 \\
   \end{array}
   $$

   Specify the pgf of $X$ and use it to find the mean and the variance of $X$.

2. The random variables $X$ and $Y$ are independent and identically distributed with pgf $P(s) = (\frac{1}{4}(1 + 3s))^4$. Find $P(X = 1)$, $E(X + Y)$, $P(X + Y = 1)$.

3. Suppose that $X$ has pgf $P_X(s) = 1/(2 - s^2)$.
   (a) Find $E(X)$, $V(X)$, $P(X = 0)$, $P(X = 1)$, $P(X = 2)$.
   (b) Find the pgf of $Y = X + 1$ and the pgf of $Z = 2X$.

4. The observed number, $Z$, of insect eggs in a clump has pmf given by:

   $$
   p_Z(z) = \frac{1}{e^\lambda - 1} \frac{\lambda^z}{z!} \quad (z = 1, 2, 3, \ldots)
   $$

   Find the pgf of $Z$ and hence find $\mu = E(Z)$ and show that $V(Z) = \mu(\lambda + 1 - \mu)$.

**Branching processes**

1. A branching process $\{X_n, n = 0, 1, 2, \ldots\}$ has $X_0 = 1$ and offspring distribution given by:

   $$
   \begin{array}{c|ccc}
   y & 0 & 1 & 2 \\
   \hline
   P(Y = y) & 0.1 & 0.4 & 0.5 \\
   \end{array}
   $$

   (a) Find $E(X_n)$.
   (b) Evaluate $P(X_n = 0)$ for $n = 1, 2, 3, 4, 5$.
   (c) Find $q$ - the probability of extinction, i.e. $q = \lim_{n \to \infty} P(X_n = 0)$.

2. Derive the equation $q_{n+1} = A(q_n)$ (refer lecture slide 446) by using the law of total probability to calculate $P(X_{n+1} = 0)$ and conditioning on the values taken by $X_1$. [Suggestion: if there are $k$ individuals in the 1st generation explain why the probability of extinction by the $(n + 1)$st generation would be $q_n^k$.]
Markov chains

1. Ghahramani Section 12.3 - Problems 1, 4, 5, 11

2. Consider two urns $A$ and $B$ which contain a total of five balls. At step $n$, a ball is selected at random from among the five balls (no matter which urn the ball is in) and placed in the other urn. Let $X_n$ denote the number of balls in urn $A$ at step $n$. Find the transition probability matrix for this Markov chain, and draw the state transition diagram.

3. A university student enrolled in the $k$th year of a three year course has probability $0.85$ of passing the year, $0.10$ of failing and having to repeat the year and $0.05$ of failing and having to quit the course. Consider a Markov chain with states $1 =$ first year, $2 =$ second year, $3 =$ third year, $4 =$ graduated and $5 =$ quit. Draw a state transition diagram and specify the transition probability matrix. Find the probability that a student who starts first year in 2005 is still studying in 2008.

4. The weather in Parktown is a Markov chain on the states \{1 = fine, 2 = overcast, 3 = wet\} with transition probability matrix given by

\[
P = \begin{bmatrix}
0.8 & 0.2 & 0 \\
0.4 & 0.5 & 0.1 \\
0.2 & 0.4 & 0.4 
\end{bmatrix}
\]

(a) If Thursday is overcast, what is the probability that
(i) Saturday is fine?
(ii) Sunday is fine?
(iii) both Saturday and Sunday are fine?

(b) What is the long-run proportion of fine days in Parktown?