Tutors to mark questions 2 and 3

1. \(X \sim \text{Bin}(n, p)\) and \(Y \sim \text{Bin}(1, 0.5)\), and \(Z = X^Y\).

(a) \[
\eta(y) = E(Z|Y = y) = E(X^y|Y = y) = \begin{cases} E(X^0|Y = 0) = E(1) = 1 \\ E(X^1|Y = 1) = E(X) = np. \end{cases}
\]

Therefore,
\[
\eta(y) = \begin{cases} 1, & y = 0 \\ np, & y = 1, \end{cases}
\]

and
\[
\eta(Y) = \begin{cases} 1, & Y = 0 \\ np, & Y = 1. \end{cases}
\]

So, \(\eta(Y)\) is a discrete random variable with pmf given by

\[
P(\eta(Y) = \mathbb{E}(X|Y) = v) = \begin{array}{cc} v & \frac{1}{2} \\ np & \frac{1}{2} \end{array}
\]

(b) \[
\nu(y) = \text{Var}(Z|Y = y) = \text{Var}(X^y|Y = y) = \begin{cases} \text{Var}(X^0|Y = 0) = \text{Var}(1) = 0 \\ \text{Var}(X^1|Y = 1) = \text{Var}(X) = np(1 - p). \end{cases}
\]

Therefore,
\[
\nu(y) = \begin{cases} 0, & y = 0 \\ np(1 - p), & y = 1, \end{cases}
\]

and
\[
\nu(Y) = \begin{cases} 0, & Y = 0 \\ np(1 - p), & Y = 1. \end{cases}
\]

So, \(\nu(Y)\) is a discrete random variable with pmf given by

\[
P(\nu(Y) = \mathbb{V}(X|Y) = v) = \begin{array}{cc} v & \frac{1}{2} \\ np(1 - p) & \frac{1}{2} \end{array}
\]

(c) \[
\mathbb{E}(Z) = \mathbb{E}(\mathbb{E}(Z|Y)) = \frac{1}{2} \times 1 + \frac{1}{2} \times np = \frac{1}{2}(1 + np).
\]
(d) 
\[ \mathbb{V}(\mathbb{E}(Z|Y)) = \mathbb{E}(\mathbb{E}(Z|Y)^2) - \mathbb{E}(\mathbb{E}(Z|Y))^2 \]
\[ = 1^2 \times \frac{1}{2} + (np)^2 \times \frac{1}{2} - \left( \frac{1}{2}(1 + np) \right)^2 \]
\[ = \frac{1}{4}(np - 1)^2. \]
\[ \mathbb{E}(\mathbb{V}(Z|Y)) = 0 \times \frac{1}{2} + np(1 - p) \times \frac{1}{2} = \frac{1}{2}np(1 - p). \]
Therefore,
\[ \mathbb{V}(Z) = \mathbb{V}(\mathbb{E}(Z|Y)) + \mathbb{E}(\mathbb{V}(Z|Y)) = \frac{1}{4}(np - 1)^2 + \frac{1}{2}np(1 - p). \]

2. We have that \( N \overset{d}{=} Pn(\Lambda) \) where \( \Lambda \overset{d}{=} \text{exp}(1) \). Then \( P(N = i) = \left( \frac{1}{2} \right)^{i+1} \implies N \overset{d}{=} \text{G}(\frac{1}{2}) \) (see Tutorial 10, Ghahramani Section 10.4, Question 9). Therefore, \( V(N) = \frac{1 - P}{np^2} = 2 \). We also have that \( \eta(\lambda) = E(N|\Lambda = \lambda) = \lambda \), therefore \( E(N|\Lambda) = \Lambda \). Also, \( \nu(\lambda) = V(N|\Lambda = \lambda) = \lambda \), therefore \( V(N|\Lambda) = \Lambda \). Thus,
\[ V(N) = E(V(N|\Lambda)) + V(E(N|\Lambda)) = E(\Lambda) + V(\Lambda) = 1 + 1 = 2. \]

3. Use the inverse transformation method to get \( X \overset{d}{=} -\frac{1}{\lambda} \log(1 - U) \) where \( U \overset{d}{=} \text{R}(0, 1) \). From Part 1 we have that
\[ E(T) = \mathbb{E}(X_1)\mathbb{E}(N) = \frac{10}{\lambda} \] (see Slide 361), and
\[ V(T) = \mathbb{V}(X_1)\mathbb{E}(N) + (\mathbb{E}(X_1))^2\mathbb{V}(N) = \frac{100}{\lambda^2} + \frac{10}{\lambda^2} = \frac{20}{\lambda^2} \] (see Slide 368).

If, for example, \( \lambda = \frac{1}{2} \), simulation should give \( E(T) \approx 20 \) and \( V(T) \approx 80 \). The empirical pdf of \( T \) is unimodal and slightly skewed to the right, see below.
4. We have that \( X \sim R(0,1), Y = \sqrt{X} = \phi(X), \) and \( \mu = \frac{1}{2} \) and \( \sigma^2 = \frac{1}{12}. \)

(a) \( \psi(x) = x^{\frac{1}{2}} \implies \psi(\mu) = \frac{1}{\sqrt{2}}. \)

\[
\psi'(x) = \frac{1}{2} x^{-\frac{1}{2}} \implies \psi'(\mu) = \frac{1}{\sqrt{2}}. 
\]

\[
\psi''(x) = -\frac{1}{4} x^{-\frac{3}{2}} \implies \psi''(\mu) = -\frac{1}{\sqrt{2}}. 
\]

\[
E(\psi(X)) \approx \frac{1}{\sqrt{2}} + \frac{1}{2} \left( -\frac{1}{\sqrt{2}} \right) \frac{1}{12} = \frac{23}{24\sqrt{2}}. 
\]

\[
V(X) \approx \left( \frac{1}{\sqrt{2}} \right)^2 \frac{1}{12} = \frac{1}{24}. 
\]

(b) \( l(x) = \frac{1}{\sqrt{2}} + \frac{1}{12} (x - \mu) \) and \( q(x) = \frac{1}{\sqrt{2}} + \frac{1}{12} (x - \mu) - \frac{1}{2\sqrt{2}} (x - \mu)^2. \)

The approximation is good as \( l(x) \) and \( q(x) \) are close to \( \psi(x) \) when \( x \in (0,1). \)

(c) The simulated values for \( E(Y) \) and \( V(Y) \) are approximately equal to \( E(\psi(X)) \) and \( V(\psi(X)) \) given in Part (a) above.