Tutorial problems:

Ghahramani: Section 10.3 - Problems 1, 6; Section 10.4 - Problems 1*, 4, 6, 9. Also the following questions:

1. If $X \overset{d}{=} R(0, \frac{\pi}{2})$ and $Z = \sin X$, find $\text{var}(Z)$ and compare this with the approximate value calculated using $\text{var}(\psi(X)) \approx \psi'(\mu)^2 \text{var}(X)$.

2. Let a random variable $X$ have the pgf

$$P_X(s) = c + 0.1(1 + s)^3 + 0.3s^5.$$  

(a) Find the constant $c$.
(b) Give the distribution of $X$ in table form.
(c) Compute the pgf of $Y = X + 2$.
(d) Use the pgf $P_X(s)$ to compute $E(X)$ and $E(X^2)$.

3. * Let $X_1, X_2, \ldots$ be independent rv’s with $E(X_j) = \mu$, $V(X_j) = \sigma^2$, $j \geq 1$, and $N \geq 0$ an integer-valued rv (with $E(N) = a, V(N) = b^2$), independent of the sequence $\{X_j\}$. Using conditional expectations and the formula $E(Y) = E[E(Y|X)]$, compute $\text{Cov}(S_N, N)$, where $S_N = \sum_{j=1}^{N} X_j$.

Additional problems: Ghahramani: Section 10.3 - Problems 4, 5; Section 10.4 - Problems 3*, 7.
NB: Solutions to the homework problems are to be given to your tutor at the start of your tutorial in the period from Monday May 22 to Friday May 26.
All four problems should be attempted. Recall that only two (randomly chosen) of them will be marked. The form and neatness of work can be considered in marking. Working and/or reasoning must be given to obtain full credit.

1. Let $X \sim \text{Bi}(n,p)$ and $Y \sim \text{Bi}(1,0.5)$ be independent random variables. Define $Z = X^Y$.
   (a) Compute the function $\eta(y) = E(Z|Y = y)$ and hence specify the rv $E(Z|Y)$.
   (b) Compute the function $\nu(y) = V(Z|Y = y)$ and hence specify the rv $V(Z|Y)$.
   (c) Use (a) to compute $E(Z)$.
   (d) Use (a) and (b) to compute $V(Z)$.

2. Consider tutorial question from Ghahramani Section 10.4, Problem 9, and let the exponentially distributed parameter be $\Lambda$. Verify that the formula
   $$V(N) = E(V(N|\Lambda)) + V(E(N|\Lambda))$$
   holds by independently calculating both sides. In your working you must explicitly identify the random variables $V(N|\Lambda)$ and $E(N|\Lambda)$.

3. Reproduce your results for Lab Exercise A, Question 2.

4. Reproduce your results for Lab Exercise B, Question 1, parts(a)-(c). Include a sketch of the plot from part (b).
In this lab you

- simulate the total claims $T$ on an insurance company in one day and check your theoretical answers for $E(T)$ and $V(T)$ against simulation estimates.

- investigate the accuracy of the approximation formulae for the mean and variance of a function of a random variable.

### Exercise A - Simulation of insurance company total claims

Suitably modified the **incomplete** Matlab m-file **Lab10ExA.m** will simulate the total claims made on an insurance company in one day. You will need to add a few lines to the program to generate the required distributions. Lab10ExA produces estimates for $E(T)$ and $V(T)$ and also plots the empirical pdf for $T$.

Let $X_1, X_2, \ldots$ be independent random variables representing the successive claims and let the number of claims in one day be $N \overset{d}{=} \text{Pn}(10)$ which is independent of $X_1, X_2, \ldots$.

We assume that $X_i \overset{d}{=} X$ (for all $i$) for some claim distribution $X$. Then $T = \sum_{i=1}^{N} X_i$ is the sum of a (random) number of random variables and represents the total claims in one day.

1. We start with the assumption that $X \overset{d}{=} \exp(\lambda)$. Using the appropriate formulae from lectures calculate the theoretical values for $E(T)$ and $V(T)$.

2. Open Lab10ExA in the m-file editor add the code required to generate the claims. Run the program for a couple of different values of $\lambda$ and compare your theoretical answers with the simulation estimates. Also comment on the shape of the empirical pdf for $T$.

3. Repeat this exercise for a claim distribution $X \overset{d}{=} R(10, 20)$.

### Exercise B - Approximations for mean and variance of functions

Let $X$ be a random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, and let $\psi(X)$ be a transformation of $X$. In general, it is **not true** that the mean and the variance of the transformation $\psi(X)$ are equal simply to the transformations of the mean and variance of $X$, respectively (an important exception is $\psi(x) = ax + b$ for which $E[\psi(X)] = \psi(E(X))$). Often it is difficult to find the exact values of $E[\psi(X)]$ and var[$\psi(X)$] (due to the fact that the integrals or sums are complicated). In lectures we derived the following approximation formulae for the mean and the variance of $\psi(X)$:

\[
E[\psi(X)] \approx \psi(\mu) + \frac{1}{2} \psi''(\mu) \sigma^2, \\
\text{var}[\psi(X)] \approx \psi'(\mu)^2 \sigma^2. \tag{1}
\]
These relations are based on the Taylor series approximations of the form

\[ \psi(X) \approx \psi(\mu) + \psi'(\mu)(X - \mu) + \frac{1}{2}\psi''(\mu)(X - \mu)^2, \]  

(2)

Refer to lecture slides 369-371. To help you better understand these formulae and test how well they work, in this lab we will apply them and verify the results using the m-files Lab10ExB1 and Lab10ExB2. Lab10ExB1 plots \( \psi(x) \) and its Taylor series approximations over a specified domain. Lab10ExB2 simulates 'nreps' observations on \( \psi(X) \) to estimate the mean and variance.

1. This first example examines the accuracy of the approximations using a simple case. Let \( X \sim R(0, 1) \) and \( Y = \sqrt{X} \) (so that \( X = \psi(X) \) with \( \psi(x) = \sqrt{x} \)).

   (a) Find the approximate values of \( E(Y) \) and \( V(Y) \).

   (b) Find the approximating functions \( l(x) \) and \( q(x) \) of \( \sqrt{x} \), given by the first 2 and 3 terms on the right-hand side of (2) respectively. (Note: \( l(x) \) is the tangent line approximation and \( q(x) \) the quadratic approximation). By adding the appropriate code to Lab10ExB1, plot \( \psi(x) \), \( l(x) \) and \( q(x) \) on the same graph, over an appropriate domain. Do you expect both approximations to be good?

   (c) Add the appropriate code to Lab10ExB2 to produce observations on \( \psi(X) \). Compare the simulation estimates with your approximations.

   (d) Verify your simulation results by calculating the exact values of \( E(Y) \) and \( V(Y) \).

   (e) Repeat this exercise for \( X \sim R(1, 2) \). Do the approximations work better or worse? Explain.

2. Let \( V = e^{-1/X} \), where \( X \sim N(5, 4) \). In this case, no exact values are readily available. Complete parts (a) through (c) of section 1 for this random variable.

3. **Challenge task:** Find an example where the approximations to \( E(\psi(X)) \) and \( V(\psi(X)) \) are both out by an order of magnitude.