Tutors to mark questions 3 and 4

1. \((X, Y) \overset{d}{=} N_2(\mu_X = 174, \mu_y = 175, \sigma^2_X = 5^2, \sigma^2_Y = 5^2, \rho = 0.7)\).

(a) 
\[
Y \mid X = 180 \overset{d}{=} N(\mu_Y + \rho \sigma_Y \left( \frac{x - \mu_X}{\sigma_X} \right), \sigma^2_Y (1 - \rho^2))
\]
\[
\overset{d}{=} N(175 + 0.7 \times 5 \times \frac{180 - 174}{5}, 5^2 \times (1 - 0.7^2))
\]
\[
\overset{d}{=} N(179.2, 12.75).
\]

(b) \(\mathbb{E}(Y) = 175, \mathbb{E}(Y \mid X = 180) = 179.2\).

(c) 
\[
P(Y > 180) = P \left( Z > \frac{180 - 175}{5} \right)
\]
\[
= P(Z > 1)
\]
\[
= 0.1587.
\]
\[
P(Y > 180 \mid X = 180) = P \left( Z > \frac{180 - 179.2}{\sqrt{12.75}} \right)
\]
\[
= P(Z > 0.224)
\]
\[
= 0.4114.
\]

2. (a) 

\[
\begin{array}{c|cccc|c}
X & 0 & 1 & 2 & 3 & 4 & p_X(x) \\
\hline
0 & 0.3 & 0.05 & 0.025 & 0.025 & 0.1 & 0.5 \\
1 & 0.18 & 0.03 & 0.015 & 0.015 & 0.06 & 0.3 \\
2 & 0.12 & 0.02 & 0.01 & 0.04 & 0.2 & 1 \\
\end{array}
\]

(b) \(P(X \leq 1, Y \leq 1) = 0.3 + 0.05 + 0.18 + 0.03 = 0.56.\) \(P(X \leq 1)P(Y \leq 1) = (0.5 + 0.3) \times (0.6 + 0.1) = 0.56.\)

(c) \(P(X + Y = 0) = P(X = 0, Y = 0) = 0.3.\)

(d) \(P(X + Y \leq 1) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 0) = 0.3 + 0.05 + 0.18 = 0.53.\)
3. \( f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = \int_{0}^{1} 2xf_Y(z-x)dx \). Note that as \( Z = X + Y, 0 < Z < 2 \) since \( 0 < X < 1 \) and \( 0 < Y < 1 \). To work out where \( f_Y(z-x) \) is nonzero we plot the inequalities \( 0 < x < 1, 0 < z < 2, \) and \( 0 < z-x < 1 \iff z-1 < x < z \) (an alternative method to do this is explained in the solutions to Tutorial 9).

From the diagram, for \( 0 < z \leq 1, f_Y(z-x) \) is nonzero if \( 0 < x < z \), and for \( 1 < z < 2, f_Y(z-x) \) is nonzero if \( z-1 < x < 1 \). Thus, for \( 0 < z \leq 1, f_Z(z) = \int_{0}^{z} 2x \times 2(z-x)dx = \frac{2}{3}z^3 \), and for \( 1 < z < 2, f_Z(z) = \int_{z-1}^{1} 2x \times 2(z-x)dx = -\frac{2}{3}z^3 + 4z - \frac{8}{3} \).
4. We have that $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$. Let $Z = X + Y$.

(a) $f_{(X,Y)}(x, y) = f_X(x)f_Y(y) = \lambda^2 e^{-\lambda(x+y)}$ where $x, y \geq 0$.

(b) Using the joint density, for $z \geq 0$,

$$F_Z(z) = P(X + Y \leq z)$$

$$= \int_0^z \int_0^{z-x} \lambda^2 e^{-\lambda(x+y)} dy \, dx$$

$$= \int_0^z \left[-\lambda e^{-\lambda(x+y)}\right]_0^{z-x} \, dx$$

$$= \int_0^z (\lambda e^{-\lambda x} - \lambda e^{-\lambda z}) \, dx$$

$$= \left[-e^{-\lambda x} - x\lambda e^{-\lambda z}\right]_0^z$$

$$= -e^{-\lambda z} - z\lambda e^{-\lambda z} - (-1 - 0)$$

$$= 1 - (z + 1)\lambda e^{-\lambda z}.$$ 

Therefore, $f_Z(z) = -(z + 1)\lambda e^{-\lambda z} \times (-\lambda) - 1 \times \lambda e^{-\lambda z} = \lambda^2 ze^{-\lambda z}$.

The convolution formula (see Slides 319–320) could also have been used. That is, for $z \geq 0$,

$$f_Z(z) = \int_{-\infty}^\infty f_X(x)f_Y(z-x) \, dx$$

$$= \int_{-\infty}^\infty (\lambda e^{-\lambda x})(\lambda e^{-\lambda(z-x)}) \, dx$$

$$= \int_0^z \lambda^2 e^{-\lambda z} \, dx$$

$$= \lambda^2 ze^{-\lambda z}.$$ 

The limits of integration are 0 and $z$ because we require $x > 0$ and $z-x > 0 \implies x < z$.

(c) $Z$ is the sum of two independent and identically distributed exponential random variables and is hence a gamma distribution. That is, $Z \sim \gamma(2, \lambda)$. The density function is as in part (b), see Slide 223.