620-201 Probability

Answers to 2002 exam paper

1. 0.5; 0.1; 0.7; 1/3; 4/7. Positive relationship.

2. (a) 0.865. (b) 0.9536.

3. (a) 7/12. (b) 4/7.

4. (a) $(1-3e^{-2})/(1-e^{-2})$. (b) $1-e^{-2}$. (c) No (as the exponential distribution is "memoryless").

5. (a) 1/4.

   (b) $F_X(x) = \begin{cases} 0 & x < 0, \\ x^4/16 & 0 \le x \le 2, \\ 1 & x > 2. \end{cases}$

   (c) 0.3125.

   (d) $c_{0.25} = 1.4142$, $c_{0.5} = \text{median} = 1.6818$, $c_{0.75} = 1.8612$.

   (e) 1.6; 8/75.

   (f) -1.05.

   (g) The rv $Y = \psi(X) \equiv X^4/16$ takes values in $(0,1)$, and for $0 < y < 1$, $\Pr(Y \le y) = \Pr(X \le 2y^{1/4}) = F_X(2y^{1/4}) = (2y^{1/4})^4/16 = y$.

   (h) Since $\psi(X) \stackrel{d}{=} R(0,1)$, $X = \psi^{-1}(U) = 2U^{1/4}$ if $U \stackrel{d}{=} R(0,1)$.

   (i) 1.0546, 1.9832, 1.9100.

6. (a) 1/2.

   (b) $f_X(x) = 2(1-x)$, $0 < x < 1$; $f_Y(y) = 2(1-y)$, $0 < y < 1$.

   (c) Not independent, but identically distributed.

   (d) $f_X(x|0.3) = 10/7$, $0 < x < 0.7$.

7. (a) 0.0934. (b) 0.4247. (c) 0.8277.

8. (a) 0.9977. (b) $2(1-e^{-6})e^{-6} = 0.0049$. (c) $F_{T_2}(t) = 1 - e^{-6t} - 6te^{-6t}$, $t \ge 0$.

9. Given $X_0 = x_0, \ldots, X_{n-1} = x_{n-1}$, the distribution of X_n is determined only by the number x_{n-1} of sick people at time $n-1$.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

10. (a) 14.95%; 31.35%; 58.96%. (b) $\boldsymbol{\pi} = (0, 0, 1)$. Can use MINITAB output to "check" if $P^n \to \Pi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ as $n \to \infty$. On the other hand, should have $\boldsymbol{\pi} = \boldsymbol{\pi} P$.

11. (a) 2,000; 20,000.
    (b) $p(x, y) = \binom{x}{y} 0.01^y 0.99^{x-y} e^{-200} 200^x / x!$, $0 \le y \le x$.
    (c)

$$\begin{aligned}
\Pr(Y = y) &= \sum_{x=y}^{\infty} \Pr(X = x, Y = y) = \sum_{x=y}^{\infty} \frac{x!}{y!(x-y)!} 0.01^y 0.99^{x-y} e^{-200} \frac{200^x}{x!} \\
&= \frac{1}{y!} 0.01^y e^{-200} 200^y \sum_{x-y=0}^{\infty} \frac{0.99^{x-y} 200^{x-y}}{(x-y)!} \\
&= \frac{2^y}{y!} e^{-200} \times e^{198} = e^{-2} \frac{2^y}{y!}, \quad y = 0, 1, 2, \ldots
\end{aligned}$$

    (d) 0.1.
    (e) $500Y$; $100^2 Y + 500^2 Y^2$; 1,000; 520,000.

12. (a) For $Y = (X_1 + X_2)/2$, $X_j \stackrel{d}{=} R(-0.1, 0.1)$, $j = 1, 2$, are independent,

$$f_Y(y) = \begin{cases} 100y + 10, & -0.1 \le y < 0, \\ 10 - 100y, & 0 \le y < 0.1, \\ 0 & \text{otherwise.} \end{cases}$$

    (b) $M_X(t) = \frac{5}{t}(e^{0.1t} - e^{-0.1t})$ $(= 1$ at $t = 0)$.
    (c) The error of the average is $Y_n = \frac{1}{n}(X_1 + \cdots + X_n)$, $X_j \stackrel{d}{=} R(-0.1, 0.1)$, $j \ge 1$, are independent, has the mgf

$$\begin{aligned}
M_{Y_n}(t) &= M_{X_1+\cdots+X_n}(t/n) = \Big[M_{X_1}(t/n)\Big]^n = \left[\frac{5n}{t}\left(e^{0.1t/n} - e^{-0.1t/n}\right)\right]^n \\
&= \left[\frac{5n}{t}\left(1 + \frac{t}{10n} + \frac{t^2}{2 \cdot 100n^2} + \frac{t^3}{6 \cdot 1000n^3} + \cdots \right.\right. \\
&\qquad \left.\left. -1 + \frac{t}{10n} - \frac{t^2}{2 \cdot 100n^2} + \frac{t^3}{6 \cdot 1000n^3} - \cdots\right)\right]^n \\
&\approx \left[\frac{5n}{t}\left(2\frac{t}{10n} + 2\frac{t^3}{6 \cdot 1000n^3}\right)\right]^n \\
&= \left[1 + \frac{t^2}{600n^2}\right]^n \approx e^{t^2/600n} = e^{(300n)^{-1} \times t^2/2},
\end{aligned}$$

which is the mgf of $N(0, 1/(300n))$. Therefore $Y_n \stackrel{d}{\approx} N(0, 1/(300n))$. The variance of the approximating distribution is $1/(300n)$.