Solutions to 620-222 exam, Semester 2 2007

Section A

1. (a) A basis is
\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}.
\]

(b) No. These matrices do not span \(W\) since they lie in the 2-dimensional subspace
\[V = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.\]

2. We have
\[
\begin{align*}
T(1) &= 1 - 1 = 0, \\
T(x) &= (x + 1) - x = 1, \\
T(x(x - 1)) &= (x + 1)x - x(x - 1) = 2x, \\
T(x(x - 1)(x - 2)) &= (x + 1)x(x - 1) - x(x - 1)(x - 2) = 3x(x - 1).
\end{align*}
\]
So the matrix is
\[
[T]_B = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

3. (a)
\[
\det(A - \lambda I) = \begin{vmatrix}
1 - \lambda & 3 & 4 \\
2 & 2 - \lambda & 4 \\
0 & 0 & -1 - \lambda
\end{vmatrix} = -(\lambda + 1)(\lambda^2 - 3\lambda - 4) = -(\lambda + 1)(\lambda + 1)(\lambda - 4)
\]
so the characteristic polynomial is \(c(X) = (x + 1)^2(x - 4)\) and the minimal polynomial \(m(X)\) is \((x + 1)(x - 4)\) or \((x + 1)^2(x - 4)\).
Now
\[
(A + I)(A - 4I) = \begin{bmatrix}
2 & 3 & 4 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
-3 & 3 & 4 \\
2 & -2 & 4 \\
0 & 0 & -5
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
so the minimal polynomial is \(m(X) = (x + 1)(x - 4)\).

(b) Yes. The matrix is diagonalizable since \(m(X)\) has only linear factors. (So all blocks in the Jordan normal form are \(1 \times 1\).)
4. From the minimal polynomial, the maximum size of Jordan blocks for 
\( \lambda = 2 \) is \( 2 \times 2 \), and the maximum size of Jordan blocks for 
\( \lambda = i \) is \( 1 \times 1 \). The sum of sizes of Jordan blocks is 5, so the possibilities are: Jordan blocks for \( \lambda = 2 \) of sizes \( 2 \times 2 \), \( 2 \times 1 \), \( 2 \times 1 \) or \( 2 \times 1 \) and the remaining diagonal entries all equal to \( i \). Thus there are 4 possible Jordan normal forms:

\[
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & i
\end{pmatrix},
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & i
\end{pmatrix},
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i
\end{pmatrix},
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i
\end{pmatrix}
\]

5. Let \( w = a(1, -1, 0) + b(0, 1, -1) \in W \) be orthogonal to \((1, 1, -1)\). Then

\[
0 = \langle (a, -a + b, -b), (1, 1, -1) \rangle = a - 2a + 2b + 3b = -a + 5b,
\]

so \( a = 5b \) and \( a = 5t, b = t \) for some \( t \in \mathbb{R} \). Thus

\[
w = 5t(1, -1, 0) + t(0, 1, -1) = t(5, -4, -1), \text{ for some } t \in \mathbb{R}.
\]

6. (a) Elements of \( G \) can have order 1, 2, 3 or 6 (by Lagrange's theorem).
We have \( 2^2 = 4, 2^3 = 8 = -1, 2^6 = 1 \), so 2 is an element of order 6. Hence \( G \) is cyclic (and 2 is a generator).

(b) \( D_3 = S_3 \) is a non-cyclic group of order 6.

7. (a) (i) \((134)(25) \cdot (12345) = (153)(24)\)
(ii) \(((12)(3456))^{-1} = (3456)^{-1}(12)^{-1} = (6543)(12)\)

(b) \((123)(4567)^k = (123)^k(4567)^k = identity \) if and only if \( k \) is a multiple of 3 and 4. So the order is \( 3 \times 4 = 12 \).

8. (a) By Lagrange’s theorem, any subgroup of \( G \) has order dividing 21.
So the possible orders are 1, 3, 7, 21.

(b) Any group of prime order is cyclic, so subgroups of order 1,3,7 are cyclic. So the only non-cyclic subgroup is \( G \) itself, with order 21.

9. (a) (1) \( G \) is closed under multiplication:
If \( A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, A' = \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} \) are in \( G \) then \( a \neq 0, a' \neq 0 \) so

\[
AA' = \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix} \in G.
\]

(2) \( G \) is closed under inversion:
If \( A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in G \) then \( A^{-1} = \begin{bmatrix} a^{-1} & -b \\ 0 & 1 \end{bmatrix} \in G. \)

Thus \( G \) is a subgroup of the matrix group \( GL(2, \mathbb{R}) \), hence is a group.
(b) For $A, A' \in G$ as above, $f(AA') = (aa')^2 = a^2(a')^2 = f(A)f(B)$ so $f$ is a homomorphism.

(c) $\text{Im}f = \{x \in \mathbb{R} : x > 0\}$ is the multiplicative group of positive reals.

$$\text{Ker}f = \left\{ \begin{bmatrix} \pm 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\}$$

10. (a) The size of each orbit divides the order of the group $G$, so each orbit contains 1, 2, 4 or 8 elements.

(b) If no orbit contains just one point, then the size of each orbit is even (2, 4 or 8). But then $|X| = \text{sum of the orbit sizes}$ would be even – contradiction.

Section B

11. (a) $T^* : V \rightarrow V$ is defined by the property:

$$\langle T^*x, y \rangle = \langle x, Ty \rangle, \text{ for all } x, y \in V.$$

(b) For all $x \in V$ we have

$$\|Tx\|^2 = \langle Tx, Tx \rangle = (T^*Tx, x)$$

$$= \langle TT^*x, x \rangle = \langle T^*x, T^*x \rangle = \|T^*x\|^2.$$

Hence $\|Tx\| = \|T^*x\|$ for all $x \in V$.

(c) $x \in \text{nullspace } (T) \iff \|Tx\| = 0 \iff \|T^*x\| = 0 \iff x \in \text{nullspace } (T^*).$

So nullspace $(T) = \text{nullspace } (T^*)$.

12. Let $A$ be an $n \times n$ complex matrix.

(a) Let $\langle \ , \ \rangle$ be the standard inner product on $\mathbb{C}^n$. If $A$ is Hermitian then $A = A^*$. Let $v \in \mathbb{C}^n$ be an eigenvector of $A$ with eigenvalue $\lambda \in \mathbb{C}$. Then $v \neq 0$ and $Av = \lambda v$. So

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Av, v \rangle = \langle v, A^*v \rangle$$

$$= \langle v, Av \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$$

Since $v \neq 0$ we have $\langle v, v \rangle \neq 0$, hence $\lambda = \bar{\lambda}$, i.e. $\lambda \in \mathbb{R}$. 

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(b) The spectral theorem states: If $A$ is a normal matrix (i.e. $A^*A = AA^*$) then there is a unitary matrix $U$ (i.e. $U^* = U^{-1}$) such that $U^*AU = D$ is a diagonal matrix whose diagonal entries are the eigenvalues of $A$. (Or: there is an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors for $A$).

If $U^*AU = D$ with $D$ real, then $A = UDU^*$ and

$$A^* = (UDU^*)^* = U^*D^*U = U^*DU = A.$$  

So $A$ is Hermitian.

13. (a) (i) \[ T(v_1) = \alpha v_1, T(v_2) = \alpha v_2 + v_1, \ldots, T(v_k) = \alpha v_k + v_{k-1}. \]

(ii) \[ [T]_{B^\prime} = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 & 0 \\ 1 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha \end{bmatrix} = J^T. \]

(iii) The matrices $J^T$, $J$ represent the same linear transformation with respect to two different bases so are similar.  
(Or: $[T]_{B^\prime} = P^{-1}[T]_B P$ where $P = P_{B^\prime \to B}$ is the transition matrix from the basis $B^\prime$ to the basis $B$. Hence $J^T = P^{-1}JP$.)

(b) By the Jordan normal form theorem, $A$ is similar to a matrix $J$ in Jordan normal form, with Jordan blocks $J_1, J_2, \ldots, J_m$. By the previous part, there is an invertible matrix $P_i$ such that $P_i^{-1}J_i P_i = J_i^T$ for each $i = 1, \ldots, m$. Then $P^{-1}JP = J^T$ where $P$ is the matrix with blocks $P_1, \ldots, P_m$ down its diagonal. Further, if $J = Q^{-1}AQ$, then

$$J^T = (Q^{-1}AQ)^T = Q^TA^T(Q^{-1})^T = Q^TA^T(Q^T)^{-1}$$

so $J^T$ is similar to $A^T$. We conclude that

$$A \sim J \sim J^T \sim A^T,$$

where $\sim$ means “is similar to”.

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14. (a) $G$ consists of

- translations by $(2\pi n, 0), n \in \mathbb{Z}$
- 180 degree rotations around $(\pi n, 0), n \in \mathbb{Z}$
- reflections in the lines $x = (n + \frac{1}{2})\pi, n \in \mathbb{Z}$
- glide reflections along the $x$-axis, translating by vectors 
  $((2n + 1)\pi, 0), n \in \mathbb{Z}$.

(b) For $(0, 0)$, orbit $= \{(n\pi, 0) : n \in \mathbb{Z}\}$, stabilizer $= \{\text{identity, 180 degree rotation around } (0, 0)\}$.

(c) $T =$ translations by $\{(2\pi n, 0) : n \in \mathbb{Z}\}$.

(d) $T$ is a normal subgroup of $G$ since it is the kernel of the homomorphism $\pi : G \to O(2)$ taking each isometry $(A, b) : x \mapsto Ax + b$ with $A \in O(2), b \in \mathbb{R}^2$ to its orthogonal part $A \in O(2)$.

15. (a) The conjugacy class of $(1)$ is $\{(1)\}$.

For a 3-cycle $\sigma$, the centralizer of $\sigma$ contains the cyclic group of order 3 generated by $\sigma$. Since

$|\text{conjugacy class}| \times |\text{centralizer}| = |G| = 12$, 

the conjugacy class of $\sigma$ contains at most 4 elements.

For $(123)$, computation using $\tau(abc)\tau^{-1} = (\tau(a)\tau(b)\tau(c))$ shows

that the conjugacy class contains $\{(123), (134), (142), (243)\}$ so this is the complete conjugacy class.

For $(132) = (123)^{-1}$, the conjugacy class consists of the inverses of the above permutations, i.e. $\{(132), (143), (124), (234)\}$.

For $(12)(34)$, the conjugacy class consists of products of two disjoint 2-cycles, and we can obtain all such permutations. So the conjugacy class is $\{(12)(34), (13)(24), (14), (23)\}$.

Hence $G$ has 4 conjugacy classes, containing 1, 3, 4 and 4 elements.

(b) A normal subgroup is closed under conjugation so must contain all conjugates of any of its elements.

(c) By (a), the conjugacy classes of $G$ contain 1,3,4,4 elements. Since 6 cannot be written as a sum of these numbers, $G$ has no normal subgroup with 6 elements.

(d) Any subgroup of order 6 would have index 2 in $G$, so would be a normal subgroup. Hence there is no subgroup of order 6.
16. (a) $G/Z$ is cyclic so is generated by some coset $aZ$ where $a \in G$. Thus for each $g \in G$, $gZ = (aZ)^n = a^nZ$ for some $n \in \mathbb{Z}$. Then $g \in a^nZ$, so $g = a^n z$ for some $z \in Z$, some $n \in \mathbb{Z}$.

Let $h = a^m z'$ be another element of $G$. Then

$$gh = a^n za^m z' = a^n a^m zz' = a^m a^n z' z = z^m z' z^n z = hg,$$

using the facts that $z \in Z$ commutes with every element of $G$, and that any two powers of $a$ commute. Hence $G$ is abelian.

(b) Let $G$ be a non-abelian group of order $p^3$ where $p$ is prime. Then the centre $Z$ of $G$ is a normal subgroup with $|Z| \geq p$ by a theorem from lectures. Hence $|Z| = p, p^2$ or $p^3$ by Lagrange’s theorem. If $|Z| = p^3$ then $G = Z$ is abelian. If $|Z| = p^2$ then $|G/Z| = p$ is prime, hence $G/Z$ is cyclic and $G$ is abelian by (a). Thus, we must have $|Z| = p$ and $G/Z$ is a non-cyclic group of order $p^2$. Hence $Z \cong \mathbb{Z}_p$ and $G/Z \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. 