620-222: Linear and Abstract Algebra
Answers to Mid-Semester Test, 12 September 2007.

(1) Consider the set $S = \{a + bi : a, b \in \mathbb{Z}\}$ with the usual operations of addition and multiplication of complex numbers. (Here, $\mathbb{Z}$ is the set of integers and $i = \sqrt{-1}$.)
(i) Is S closed under multiplication? (ii) Is S a field?
Give brief reasons for your answers.

Solution:
(i) This is closed under multiplication. If $a_1 + b_1 i, a_2 + b_2 i \in S$ with $a_1, b_1, a_2, b_2 \in \mathbb{Z}$, then $(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i \in S$ since $a_1 a_2 - b_1 b_2$ and $a_1 b_2 + a_2 b_1$ are in $\mathbb{Z}$.
(ii) This is not a field. For example, 2 has no multiplicative inverse.

(2) Let U be the real vector space of all upper triangular 2×2 real matrices:

$$U = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

a spanning set for U? Give brief reasons for your answer.

Solution: We try to find $x, y, z \in \mathbb{R}$ such that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or $x + z = a, b = 2y, -x + z = c$. This can be solved for all $a, b, c \in \mathbb{R}$:
$y = \frac{1}{2}b, x = \frac{1}{2}(a - c), z = \frac{1}{2}(a + c)$. Hence the matrices span U.

(3) Let $V = \mathcal{P}_2(\mathbb{R})$ denote the real vector space of polynomials with real coefficients having degree at most 2. Let $T : V \to V$ denote the linear transformation given by $T(p(x)) = p(x + 1)$ and let $\mathcal{B}$ be the standard basis $\{1, x, x^2\}$ for V. Find the matrix of T with respect to the basis $\mathcal{B}$.

Solution: $T(1) = 1$, $T(x) = x + 1$, $T(x^2) = (x + 1)^2 = x^2 + 2x + 1$. Hence T has matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(4) Show that if V and W are 5-dimensional subspaces of the complex vector space $\mathbb{C}^9$, then $V \cap W$ contains a non-zero vector.

Solution: We have $\dim(V \cap W) + \dim(V + W) = \dim V + \dim W$. Since $V + W$ is a subspace of $\mathbb{C}^9$, its dimension is at most 9. We conclude that $\dim(V \cap W) \geq 5+5-9 = 1$ and so the intersection contains a non-zero vector.

(5) (i) Find the minimal polynomial of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

(ii) Is the matrix diagonalizable?
Give brief reasons for your answers.

Solution:
(i) The matrix has characteristic polynomial $(X-1)^2(X+3)$. The minimal polynomial $m(X)$ divides this and has every eigenvalue as a root, so $m(X)$ is either $(X-1)(X+3)$ or $(X-1)^2(X+3)$. But

$$(A-I)(A+3I) = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the minimal polynomial is $m(X) = (X-1)(X+3)$.
(ii) Since $m(X)$ has no repeated factors, the matrix is diagonalizable. (All Jordan blocks are 1×1.)

(6) A complex matrix A has characteristic polynomial $(X-4)(X-2i)^3$ and its $\lambda = 2i$ eigenspace has dimension 2. Find the Jordan Normal Form of the matrix. Give a brief explanation.

Solution: The JNF has one 4 and three $2i$ entries on the diagonal, and there are two Jordan blocks for $\lambda = 2i$. Hence the JNF is

$$\begin{bmatrix} 2i & 1 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(7) Vectors v, w in a complex inner product space satisfy $||v|| = 1, ||w|| = 3, (v, w) = 1+2i$. Find $||v + iw||$.

Solution: Using the inner product properties we have

$$\begin{aligned} ||v+iw||^2 &= (v+iw, v+iw) \\ &= (v,v) + i(w,v) - i(v,w) + (w,w) \\ &= 1^2 + i(1-2i) - i(1+2i) + 3^2 \\ &= 1+4+9 = 14 \end{aligned}$$

so $||v + iw|| = \sqrt{14}$.

(8) Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 0, 2)$ and $(2, -1, 1)$. Find a basis for the orthogonal complement of W using the dot product as inner product.

Solution: $W^\perp$ is the set of all solutions of $x + 2z = 0, 2x - y + z = 0$ since any vector in $W^\perp$ is orthogonal to the basis for W. But then we find $x = -2z, y = -3z$ and so a basis is the single vector $(-2, -3, 1)$.
(Alternatively: the cross product $(1, 0, 2) \times (2, -1, 1) = (2, 3, -1)$ is a basis.)

(9) (i) Find the adjoint of the matrix $A = \begin{bmatrix} 1 & i \\ 1 & 2+i \end{bmatrix}$.

(ii) Is A Hermitian? (iii) Is A normal?

Give brief reasons for your answers.

Solution:

(i)

$$A^* = \overline{A}^T = \begin{bmatrix} 1 & 1 \\ -i & 2-i \end{bmatrix}$$

(ii) Since $A^* \neq A$, A is not Hermitian.

(iii)

$$AA^* = \begin{bmatrix} 2 & 2+2i \\ 2-2i & 6 \end{bmatrix} = A^*A,$$

so A is normal.

(10) Let $\mathbb{R}^\infty$ be the real vector space of all real sequences. Find all eigenvalues and eigenvectors for the "backward shift" operator $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by

$$T(a_1, a_2, a_3, a_4, \ldots) = (a_2, a_3, a_4, \ldots).$$

Solution: Assume $T(a_1, a_2, a_3, a_4, \ldots) = \lambda(a_1, a_2, a_3, a_4, \ldots)$. Then

$$(a_2, a_3, a_4, \ldots) = \lambda(a_1, a_2, a_3, a_4, \ldots),$$

so

$$a_2 = \lambda a_1, a_3 = \lambda a_2 = \lambda^2 a_1, \ldots, a_{n+1} = \lambda^n a_1.$$

Hence every real number λ is an eigenvalue of T, and the corresponding eigenvectors of T are the geometric sequences: $a_1(1, \lambda, \lambda^2, \lambda^3, \ldots)$.