1. We work, as usual, with the augmented matrix and use row reduction, but all arithmetic is in $\mathbb{Z}_7$. We write $n$ as an abbreviation for $[n]_7$.

\[
\begin{pmatrix} 2 & 4 & 1 & 6 \\ 3 & 5 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 3 \\ 3 & 5 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 6 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{pmatrix}
\]

Note that at the first step we multiplied row 1 by $[2]_7^{-1} = [4]_7$, and at the third step we multiplied row 2 by $[6]_7^{-1} = [6]_7$.

Since the leading entries occur in columns 1 and 2 we can take $z = t$ as a free parameter and solve for $y$ and $x$:

\[
x = 5 - 5t = 5 + 2t, \quad y = 6 - 3t = 6 + 4t, \quad z = t
\]

where $z \in \mathbb{Z}_7$ is arbitrary.

2. (a) Let $F = \{a + b\sqrt{2} + ci + d\sqrt{2}i : a, b, c, d \in \mathbb{Q}\}$. Then $F$ is a subset of the field $\mathbb{C}$, so field properties (1), (4), (5), (8), (9) automatically hold in $F$. So we only need to check the subfield properties:

(i) $F$ is closed under addition and multiplication
(ii) $F$ contains 0 and 1
(iii) for each $x$ in $F$, $-x \in F$
(iv) for each $x \neq 0$ in $F$, $x^{-1} \in F$.

(i) Let $x = a + b\sqrt{2} + ci + d\sqrt{2}i$ and $x' = a' + b'\sqrt{2} + c'i + d'\sqrt{2}i$ be in $F$ with $a, b, c, d, a', b', c', d' \in \mathbb{Q}$. Then

\[
x + x' = (a + a') + (b + b')\sqrt{2} + (c + c')i + (d + d')\sqrt{2}i \in F,
\]

since $a + a', b + b', c + c', d + d' \in \mathbb{Q}$. Similarly, by expanding the product,

\[
xx' = (a + b\sqrt{2} + ci + d\sqrt{2}i)(a' + b'\sqrt{2} + c'i + d'\sqrt{2}i) = A + B\sqrt{2} + Ci + D\sqrt{2}i,
\]

where $A = aa' + 2bb' - cc' - 2dd', B = ab' + ba' - cd' - dc', C = ac' + 2bd' + ca' + 2db', D = ad' + bc' + cb' + da' \in \mathbb{Q}$. Hence $xx' \in F$ and $F$ is closed under addition and multiplication.

(ii) $F$ contains 0 = 0 + 0\sqrt{2} + 0i + 0\sqrt{2}i and 1 = 1 + 0\sqrt{2} + 0i + 0\sqrt{2}i.

(iii) If $x = a + b\sqrt{2} + ci + d\sqrt{2}i \in F$, then $-x = -a - b\sqrt{2} - ci - d\sqrt{2}i \in F$.

(iv) Let $x = a + b\sqrt{2} + ci + d\sqrt{2}i \in F$ be non-zero. Write $x = A + Bi$, where $A = a + b\sqrt{2}, B = c + d\sqrt{2}$ and $a, b, c, d \in \mathbb{Q}$. Since $x \neq 0$, it has an inverse in $\mathbb{C}$:

\[
x^{-1} = \frac{1}{A + Bi} = \frac{A - Bi}{(A + iB)(A - iB)} = A - Bi \quad \frac{1}{A^2 + B^2}
\]
Now the numerator $A - Bi = (a + b\sqrt{2}) - (c + d\sqrt{2})i$ is in $F$, and the denominator is $A^2 + B^2 = (a + b\sqrt{2})^2 + (c + d\sqrt{2})^2 = p + q\sqrt{2}$, where

\[ p = a^2 + 2b^2 + c^2 + 2d^2, \quad q = 2(ab + cd) \in \mathbb{Q}. \]

Thus

\[ \frac{1}{A^2 + B^2} = \frac{1}{p + q\sqrt{2}} = \frac{p - q\sqrt{2}}{p^2 - 2q^2} \in F. \]

Note that the denominator here is non-zero since $p^2 - 2q^2 = 0$ implies $p = q = 0$ (and $x = 0$) as $\sqrt{2}$ is irrational. Since $F$ is closed under multiplication, follows that $x^{-1} \in F$ whenever $0 \neq x \in F$.

From (i)–(iv) we conclude that $F$ is a field (in fact, it is a subfield of $\mathbb{C}$).

(b) Consider the set $S = \{1, \sqrt{2}, i, \sqrt{2}i\}$. Clearly each element of $F$ can be written $a.1 + b.\sqrt{2} + c.i + d.\sqrt{2}i$ with $a, b, c, d \in \mathbb{Q}$ so $S$ spans $F$. Now suppose $a.1 + b.\sqrt{2} + c.i + d.\sqrt{2}i = 0$ with $a, b, c, d \in \mathbb{Q}$. Equating real and imaginary parts gives:

\[ a + b\sqrt{2} = 0 \text{ and } c + d\sqrt{2} = 0. \]

If $b \neq 0$ then the first equation gives $\sqrt{2} = -a/b \in \mathbb{Q}$ contradicting the fact that $\sqrt{2}$ is irrational. Hence $b = 0$ and $a = -b\sqrt{2} = 0$. Similarly $c = d = 0$. So $S$ is linearly independent, hence a basis for $F$ over $\mathbb{Q}$. Thus $\dim_{\mathbb{Q}} F = 4$.

(c) Let $\alpha = \sqrt{2} + i$. Now $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ are 5 elements in a vector space $F$ of dimension 4 over $\mathbb{Q}$, so must be linearly dependent over $\mathbb{Q}$. Hence there exist $c_0, c_1, c_2, c_3, c_4 \in \mathbb{Q}$, not all zero, such that $c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4 = 0$. (In fact, $\alpha^2 - 2\alpha^2 + 9 = 0$, but this is not needed.)

3. (a) $f$ is a linear transformation since

\[ f(X + Y) = A(X + Y) - (X + Y)A = AX + AY -XA - YA \]

\[ = (AX - XA) + (AY - YA) = f(X) + f(Y) \]

and

\[ f(\alpha X) = A(\alpha X) - (\alpha X)A = \alpha(AX -XA) = \alpha f(X) \]

for all $X, Y \in V$ and all $\alpha \in \mathbb{C}$.

(b) We have

\[ f\left(\begin{bmatrix} a \\ c \\ \hline b \\ d \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ \hline c \\ d \end{bmatrix} - \begin{bmatrix} a \\ b \\ \hline c \\ d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -b & 0 \\ a - d & b \end{bmatrix}. \]

Applying $f$ to the vectors in the standard basis $B$ gives

\[ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

Writing these out as linear combinations of the basis vectors gives the columns of the matrix of $f$ with respect to the basis $B$:

\[ M = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \]
(c) From (\(*\)), the nullspace of \(f\) consists of matrices with \(b = 0\), \(a = d\) and \(c\) arbitrary, i.e.

\[
N = \left\{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} : a, c \in \mathbb{C} \right\}
\]

and the range is

\[
R = \left\{ \begin{bmatrix} -b & 0 \\ a & b \end{bmatrix} : a, b \in \mathbb{C} \right\}.
\]

Now we can choose one basis vector corresponding to each parameter in these subspace descriptions. Hence \(f\) has nullity = \(\dim N = 2\) and rank = \(\dim R = 2\).

(d) The eigenvalues of \(f\) are the same as the eigenvalues of the matrix \(M\) so satisfy the characteristic equation

\[
0 = \det(M - \lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 0 & -\lambda \end{vmatrix} = (-\lambda) \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 0 & -\lambda \end{vmatrix}
\]

\[
= (-\lambda)^2 \begin{vmatrix} -\lambda & -1 \\ 0 & -\lambda \end{vmatrix} = \lambda^4,
\]

expanding by cofactors (for example using row 2 then row 1). Thus the only eigenvalue is \(\lambda = 0\). The corresponding eigenspace is the nullspace \(N\) of \(f\) calculated above, and one possible basis is

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.
\]

(This is a basis since every matrix in \(N\) can be written uniquely as a linear combination of these two matrices.)

(e) From (e), the characteristic polynomial is \(c(X) = X^4\). Since the minimal polynomial divides \(c(X)\), the minimal polynomial must be \(X, X^2, X^3\) or \(X^4\). Substituting \(M\) into these gives

\[
M \neq 0, M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0, M^3 = 0.
\]

So the minimal polynomial is \(m(X) = X^3\).

From \(c(X)\) we see that the Jordan normal form has all diagonal entries 0, and from \(m(X)\) we see that the maximum size of Jordan blocks is \(3 \times 3\). So the Jordan normal form has one \(3 \times 3\) block and one \(1 \times 1\) block, i.e.

\[
J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]
(f) (Bonus) We need to find a basis \( \{v_1, v_2, v_3, v_4\} \) such that \( f(v_1) = 0, \)
\( f(v_2) = v_1, f(v_3) = v_2, f(v_4) = 0. \) Note that \( f^2(v_3) = v_1 \neq 0 \) so we need to find \( v_3 \) such that \( f^2(v_3) \neq 0. \) Then we can take \( v_2 = f(v_3), v_1 = f^2(v_3). \) Finally we need to choose \( v_4 \) not a multiple of \( v_1 \) such that \( f(v_4) = 0. \) One possibility is to take:

\[
\begin{align*}
v_3 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_2 = f(v_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, v_1 = f(v_2) = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{align*}
\]

(But this is \textbf{not} the only possibility!)