

Department of Mathematics and Statistics
620-222: Linear and Abstract Algebra 2007
Assignment 2

Your solutions should be handed in by 5.15pm on Wednesday October 24th.
Please give complete explanations in all questions.

1. Let p be a prime number.

(a) Show that

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}_p \right\}$$

is a subgroup of $GL(3, \mathbb{Z}_p)$ using matrix multiplication as the operation.

(b) Show that G is a non-abelian group of order p^3 .

(c) Show that the function $f : G \rightarrow \mathbb{Z}_p \times \mathbb{Z}_p$ defined by

$$f \left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right) = (a, b)$$

is a homomorphism from G to the additive group $\mathbb{Z}_p \times \mathbb{Z}_p$.

(d) Find the kernel and image of f .

(e) Show that $\ker f$ is the centre Z of G .

(f) Use parts (d) and (e) to identify the quotient group G/Z .

(g) Consider the group G obtained when $p = 2$.

(i) Find the number of elements of order 1, 2, 4 and 8 in G .

(ii) There are exactly two non-abelian groups of order 8 up to isomorphism: the dihedral group D_4 and the quaternion group Q_8 with multiplication tables as shown below. Which of these groups is isomorphic to G ? Explain your answer.

D_4 Dihedral group: $a^4 = b^2 = (ab)^2 = 1$

	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

Q_8 Quaternion group: $a^4 = 1, a^2 = b^2, ba = a^3b$

	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	1	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	1
a^2b	a^2b	ab	b	a^3b	1	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	1	a^3	a^2

2. (a) Let H be a normal subgroup of a group G , and let K be a subgroup of H . Prove that if H is cyclic, then K is a normal subgroup of G .
- (b) Give an example of a normal subgroup H of a group G and a subgroup K of H such that K is not normal in G .
3. The aim of this question is to investigate the group G of all rotational symmetries of a cube X in \mathbb{R}^3 .
 - (a) Make yourself a cube. A prize will be awarded for the best model brought to class on October 24th. (Hint: cardboard is a suitable material.)
 - (b) Consider the action of G on the cube X . Describe the stabilizer and orbit of (i) a vertex, (ii) the midpoint of a face, (iii) the midpoint of an edge.
 - (c) Deduce that $|G| = 24$. (The previous question gives three ways of showing this.)
 - (d) Consider the action of G on the set D consisting of the four "long diagonals" of the cube. (Each of these diagonals passes from a vertex through the centre of the cube to the opposite vertex.) This defines a homomorphism $\sigma : G \rightarrow \text{Sym}(D)$ into the group $\text{Sym}(D)$ of all permutations of D , and by numbering the four diagonals we can identify the group $\text{Sym}(D)$ with S_4 .
 - (i) Show that any symmetry which maps each diagonal in D to itself must either fix each vertex of the cube or else it must send each vertex to the opposite vertex. (Note that a diagonal may be mapped so that its end points are interchanged.)
 - (ii) Show that, in the first case of the previous part, the symmetry must be the identity and that the second case is not possible if the symmetry is a rotation.
 - (iii) Deduce that the kernel of σ is the trivial subgroup and so that σ is an injection (i.e. one to one).
 - (iv) Use part (c) to conclude that σ is an isomorphism, hence $G \cong S_4$.