COPY

The University of Melbourne

Department of Mathematics and Statistics

Semester Two Assessment 2004

620–222 Algebra

Examination duration: Three hours
Reading time allowed: Fifteen minutes

This paper has five pages in total including this page.

Authorized Materials: No materials are authorized. Students entering the examination venue with notes or printed material related to the subject, calculators or computers, or mobile phones, should stand in their place immediately and surrender these to an invigilator before the instruction to commence writing is given.

Instructions to Invigilators: No special materials are to be supplied. The paper may be removed at the conclusion of the examination.

Instructions to Students: This examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that candidates attempt the questions in Section A before attempting those in Section B. It is possible to pass the examination on marks from Section A alone. All questions may, however, be attempted.

SECTION A

Question 1

Decide whether the following polynomials form a spanning set in the complex vector space $P_2(\mathbf{C})$ of all polynomials of degree at most 2 with complex coefficients. Does this collection of polynomials form a basis for $P_3(\mathbf{C})$? Give brief reasons for your answers.

$\{x^2 + i, \quad x - 1, \quad x^2 + x, \quad 2i - 1\}$

(6 marks)

Question 2

Let V be a real vector space with basis $\mathbf{B} = \{v_1, v_2, v_3\}$ and let $f : V \to V$ be the linear transformation satisfying

$$f(v_1) = 2v_2 + v_3, f(v_1 + 2v_2) = v_3, f(v_3 + 3v_1) = v_2.$$

Find the matrix of f relative to the basis $\mathbf{B}$.

(6 marks)

Question 3

Show that the complex matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is Hermitian. Is this matrix diagonalisable? Give a brief reason for your answer.

(6 marks)

Question 4

A 6×6 complex matrix X has minimal polynomial $m(X) = (X + 2)^3$. Determine all the possible Jordan normal forms for the matrix X (up to reordering of the Jordan blocks).

(6 marks)

Question 5

Let W be the real vector space of all linear transformations $g : \mathbf{R^3} \to \mathbf{R}$. So the sum and scalar multiplication are given by $(g + h)(v) = g(v) + h(v)$ and $(\lambda g)(v) = \lambda(g(v))$, for $g, h \in W$ and $\lambda \in \mathbf{R}$. Define an inner product on W by $(g, h) = g(e_1)h(e_1) + g(e_2)h(e_2) + g(e_3)h(e_3)$, where $\{e_1, e_2, e_3\}$ is the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ for $\mathbf{R^3}$ (You do not need to check this is an inner product).

Let V be the subspace of W spanned by the transformations f_1, f_2, where $f_1(e_1) = f_1(e_2) = 1, f_1(e_3) = 0$ and $f_2(e_1) = 0, f_2(e_2) = f_2(e_3) = -1$. Find the orthogonal complement of V relative to this inner product.

(6 marks)

Question 6

(a) Calculate the following permutation as a products of disjoint cycles $(135)(24) \cdot (12345)$

(b) Find all conjugates of $(12)(34)$ in the symmetric group S_4 of permutations of $1, 2, 3, 4$.

(6 marks)

Question 7

(a) Show that the collection of real 2×2 matrices A of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where a, b are real numbers with $a \neq b, a \neq -b$, forms a group G under matrix multiplication.

(b) Prove that the map $\phi : G \to \mathbf{R}^*$ given by $\phi(A) = a + b$ is a homomorphism from G to the multiplicative group $\mathbf{R}^*$ of non zero real numbers.

(c) Find the kernel and the image of ϕ.

(6 marks)

Question 8

The set $\{1, 3, 5, 9, 11, 13\}$ forms a group G under multiplication modulo 14.

(a) List the possible orders of subgroups of G.

(b) Find the orders of all elements of the group G.

(c) Is this group cyclic? Give a reason for your answer.

(6 marks)

Question 9

Consider the set R of points (x, y) in the plane satisfying either $1 \leq |x| \leq 2$ and $y = 0$, or or $x = 0$ and $1 \leq |y| \leq 2$.

(a) Draw R and describe the group G of its symmetries.

(b) For the point $(1, 0)$ in R, find its orbit and stabiliser.

(c) Explain how the orbit/stabiliser theorem connects G and the orbit and stabiliser in (b).

(6 marks)

Question 10

Let G be a group of order 10.

(a) Suppose that G has a subgroup H of order 5. Explain why H must be normal in G.

(b) If $G = Z_2 \times Z_5$ explain why G is isomorphic to Z_{10}.

(c) Give an example of a group of order 10 which is not cyclic.

(6 marks)

SECTION B

Question 11

Let V be a complex finite dimensional inner product space and let $f : V \to V$ be a linear transformation satisfying $f^* = f^{-1}$.

(a) State the spectral theorem and deduce that there is an orthonormal basis of V consisting of eigenvectors of f.

(b) Show that $(f(u), f(v)) = (u, v)$, for all vectors $u, v \in V$, where (a, b) is the inner product of vectors in V.

(c) Show that every eigenvalue of f has absolute value 1.

(d) Give an example to show that the result in (a) can fail if V is a real inner product space. (Hint: Consider the case $V = \mathbf{R}^2$).

(10 marks)

Question 12

Suppose that a linear transformation f from a real vector space to itself has characteristic polynomial $X^2(X+1)^2(X-1)^2$.

(a) Describe the possible minimal polynomials of f.

(b) Hence list the possible Jordan canonical forms of f. (You can ignore all reorderings of Jordan blocks).

(c) Explain why the linear transformation f cannot be invertible..

(10 marks)

Question 13

(a) Let A be an $n \times n$ complex Hermitian matrix. Define a product on $\mathbf{C}^n$ by $(X, Y) = XAY^*$, where $X, Y \in \mathbf{C}^n$ are written as row vectors. Show that this is an inner product, if all the eigenvalues of A are positive real numbers. (Hint: write $A = UDU^*$, where D is diagonal).

(b) Show that if $A = B^*B$, where B is any invertible $n \times n$ complex matrix, then A is a Hermitian matrix and all the eigenvalues of A are real and positive. (Hint: If X is an eigenvector of A, written as a column vector, consider the product X^*AX.)

(10 marks)

Question 14

Consider the group $G = S_3 \times S_3$ which is a direct product of two copies of the symmetric group S_3.

(a) Find the orders of all the elements of G.

(b) Find two subgroups of G of order 6 which are not isomorphic.

(c) Describe a normal subgroup of G different from the trivial subgroups ($\{((1),(1))\}$ and G) and describe a subgroup of G which is not normal.

(10 marks)

Question 15

Consider the symmetric group S_4 acting on the four numbers $\{1,2,3,4\}$. Consider the three ways of dividing these numbers into two pairs, namely $P_1 = \{1,2\},\{3,4\}$, $P_2 = \{1,3\},\{2,4\}$, $P_3 = \{1,4\},\{2,3\}$.

(a) Prove that there is a homomorphism from S_4 onto S_3 by using the action of S_4 on $\{1,2,3,4\}$ to give an action of S_4 on the set of three objects $\{P_1, P_2, P_3\}$

(b) Describe the elements of the kernel of this homomorphism and explain why this subgroup is isomorphic to the group D_2 of symmetries of a rectangle.

(10 marks)

Question 16

Consider the infinite pattern of symbols

...ZZZZZZ...

(a) Describe the full group G of symmetries of this pattern.

(b) Find the stabiliser H of one of the symbols **Z**.

(c) Find a maximal normal subgroup of translations T in G and explain why the quotient group G/T is isomorphic to the stabiliser subgroup H.

(10 marks)

This is the end of the examination. Please check your working carefully.