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The University of Melbourne

Semester 2 Assessment, 2005

Department of Mathematics and Statistics

620-222 Linear and Abstract Algebra

Instructions to Students:
The examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that candidates attempt the questions in Section A before trying those in Section B. It is possible to pass the examination on marks from Section A alone. All questions may be attempted.

The total number of marks for this paper is 120.

Please give complete explanations in all questions, and give careful statements of any results from the notes or lectures that you use.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorized materials:
Numerical calculators, pens, rubbers, and rulers are authorized. No other materials are authorized. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Reproduction of question paper: After the examination, this question paper may be reproduced and lodged in the Baillieu Library.

Section A

1. Consider the polynomials $i + x, x + (1+i)x^2, x^2 - 2ix^3$ in the complex vector space $\mathcal{P}_3$ of all polynomials of degree at most 3 with complex coefficients.

   (a) Are the polynomials linearly independent?

   (b) Do the polynomials form a basis for $\mathcal{P}_3$?

   Give brief explanations.

   6 marks

2. Let V be a real vector space with bases $\mathcal{B} = \{v_1, v_2\}$ and $\mathcal{B}' = \{v_1 - v_2, v_1 + 2v_2\}$. Let $f : V \to V$ be a linear transformation satisfying

$$f(v_1) = v_1 + 2v_2, \quad f(v_2) = 3v_1.$$

   (a) Find the matrix of f with respect to the basis $\mathcal{B}$.

   (b) Find the matrix of f with respect to the basis $\mathcal{B}'$.

   (c) Deduce the eigenvalues of f.

   6 marks

3. (a) Find the minimal polynomial for the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (b) Is the matrix diagonalizable? Give brief reasons.

   6 marks

4. A 6×6 complex matrix has $m(X) = (X-4)^2(X+i)^2$ as minimal polynomial . Determine all the possible Jordan normal forms for the matrix (up to rearranging the Jordan blocks).

   6 marks

5. (a) Is the matrix $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ any of Hermitian or unitary or normal?

   (b) Is the matrix in (a) diagonalisable? Given a brief reason for your answer.

   (c) Describe a 2×2 matrix which is both Hermitian and unitary, but not the identity matrix.

   6 marks

6. The set $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ forms a subgroup of the group of permutations S_4.

   (a) Find the order of each element in the group H.
   (b) Is the group H cyclic?
   (c) Is the group H Abelian?

   Give brief explanations.

   6 marks

7. Let G be a group of order 27.

   (a) What are the possible orders of subgroups of G?
   (b) Suppose that a subgroup H of G is different from G and is not cyclic. Find the order of H.

   Give brief explanations.

   6 marks

8. Let G be the direct product $\mathbb{Z}_2 \times \mathbb{Z}_3$.

   (a) Find an element of order 6 in G.
   (b) Construct a mapping from G to $\mathbb{Z}_6$, which is an isomorphism. Give brief reasons for your answer.

   6 marks

9. Let G be the multiplicative group of 2×2 real matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, where $a \neq 0$. Define $f : G \to \mathbb{R} \setminus \{0\}$ by $f(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}) = a$.

   (a) Show that f is a homomorphism to the group of non-zero real numbers under multiplication.
   (b) Find the image and kernel of f.

   6 marks

10. Let X be the hexagon in $\mathbb{R}^2$ with vertices $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(-1, 0)$, $(\frac{1}{2}, -\frac{1}{2})$,$(-\frac{1}{2}, -\frac{1}{2})$.

    (a) Sketch X and list the elements of the group G of all symmetries of X, taking X to itself by isometries.
    (b) For the action of G on X, find the stabilizer and orbit of
        i. the vertex $(1, 0)$.
        ii. the vertex $(\frac{1}{2}, \frac{1}{2})$.

    6 marks

Section B

11. (a) Suppose that A is a complex $n \times n$ matrix and $p(x)$ is a polynomial with complex coefficients. Prove that if λ is any eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$.

    (b) Conversely use the Jordan normal form of A to prove that if γ is an eigenvalue of $p(A)$, then $\gamma = p(\lambda)$, for some eigenvalue λ of A.

    10 marks

12. Let V be a finite dimensional real inner product space and let V^* denote the space of linear transformations from $V \to \mathbb{R}$.

    (a) Show that V^* has an inner product, which is defined by $(f, g) = f(e_1)g(e_1) + \cdots + f(e_n)g(e_n)$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V and $f, g \in V^*$.

    (b) Show that there is a linear transformation T from V to V^* given by $T : w \to f$ where $f(u) = (u, w)$, for $w \in V$ and $f \in V^*$.

    (c) Prove that $\{e_1^*, \ldots, e_n^*\}$ is an orthonormal basis for V^*, where $e_i^*(e_j) = 0$, for $i \neq j$ and $e_i^*(e_i) = 1$, for all i, j between 1 and n.

    (d) Prove that $T(e_i) = e_i^*$ and hence that T is an isometry.

    10 marks

13. Let V be a finite dimensional complex inner product space and let $f : V \to V$ be a linear transformation satisfying $f^* = f^{-1}$.

    (a) State the spectral theorem, and deduce that there is an orthonormal basis of V consisting of eigenvectors for f.

    (b) Show that $(f(v), f(w)) = (v, w)$ for all $v, w \in V$.

    (c) Show that every eigenvalue of f has absolute value 1.

    (d) Give an example to show that the result of (a) can fail if V is a real inner product space. [Hint: consider functions $f : \mathbb{R}^2 \to \mathbb{R}^2$].

    10 marks

14. Let G be the group of 2×2 matrices: $\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\}$. Let $\{+1, -1\}$ be a group under multiplication. Define a homomorphism $\phi : G \longrightarrow \{+1, -1\}$, by $\phi(A) = \det A$. (Here $\det A$ is the determinant of the matrix A).

   (a) Find the kernel and image of ϕ.
   (b) Find a normal subgroup H of G so that the quotient group G/H is isomorphic to $\mathbb{Z}_2$. Justify your answer.
   (c) Determine if the group G is cyclic or not.

   10 marks

15. For any isometry $f : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ of the Euclidean plane, let

$$\text{Fix}(f) = \{x \in \mathbb{E}^2 : f(x) = x\}$$

   denote the fixed point set of f.

   (a) The non-identity isometries of $\mathbb{E}^2$ are of four types: rotations, reflections, translations and glide reflections. Describe the fixed point set for each type.
   (b) Describe all the symmetries of the infinite pattern:

$$\cdots \mathbf{HHHHHHHH} \cdots$$

   10 marks

16. Let $\mathbb{H}$ be the quaternions, i.e the four dimensional real vector space with basis $1, i, j, k$ where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$.

   (a) Show that $\mathbb{H} \setminus \{0\}$ forms a group under multiplication. [Hint; Use the same trick as for complex numbers, namely define $\bar{u} = a_0 - ia_1 - ja_2 - ka_3$ where $u = a_0 + ia_1 + ja_2 + ka_3$ is an arbitrary quaternion, with a_0, a_1, a_2, a_3 are real numbers. Then check that $u\bar{u} = \bar{u}u = a_0^2 + a_1^2 + a_2^2 + a_3^2$].
   (b) Define a map $f : \mathbb{H} \setminus \{0\} \longrightarrow \mathbb{R}^+$ by $f(u) = u\bar{u}$, where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers. Prove that f is a homomorphism.
   (c) Deduce that the unit sphere S^3 in $\mathbb{R}^4$ has the structure of a group. [Hint; $S^3 = \{(a_0, a_1, a_2, a_3) : a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\}$].

   10 marks