Isometries of Euclidean space (§2·8·4)

We now look at the groups that arise in Euclidean geometry.

Let $\mathbb{E}^n$ = "Euclidean n-space" be $\mathbb{R}^n$ together with the usual Euclidean notion of distance:

$$d(\boldsymbol{x}, y) = \|x-y\| = \sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$$

for $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ in $\mathbb{R}^n$.

DEFINITION: An isometry of $\mathbb{E}^n$ is a distance preserving function $f:\mathbb{R}^n\to\mathbb{R}^n$, i.e. $d(x,y) = d(f(x), f(y))$ for all $x,y\in\mathbb{R}^n$.

Examples:

(a) rotations, reflections & translations are isometries of $\mathbb{E}^2$.

(b) If $A\in O(n)$ is an orthogonal matrix (ie. $A^TA=I$), then the linear transformation $A:\mathbb{R}^n\to\mathbb{R}^n$, $A(x)=Ax$ is an isometry.

[Proof: We know

$$Av\cdot Aw = A^TAv\cdot w = v\cdot w \quad (A^*=A^T)$$

for all $v,w\in\mathbb{R}^n$. Taking $v=w$ gives $\|Av\|=\|v\|$ for all $\mathbf{v}$.

Hence $\|Ax-Ay\| = \|A(x-y)\| = \|x-y\|$, for all $x,y\in\mathbb{R}^n$.]

(c) If $b\in\mathbb{R}^n$, then translation by b is an isometry: $t_b(x)=x+b$.

Further $t_b^{-1}=t_{-b}$ (Check!)

(d) Compositions of isometries are isometries. (Exercise.)

In fact, every isometry of $\mathbb{E}^n$ is a composition of translations and orthogonal transformations.

(2·8·6)
LEMMA: Let f be an isometry of $\mathbb{E}^n$ fixing the origin $\underline{0}$. Then f is a linear transformation:

$f(x) = Ax$, where $A\in O(n)$ is an orthogonal matrix.

Proof: See Notes (Lemma 2·8·6).

(2·8·7)
COROLLARY: Every isometry f of $\mathbb{E}^n$ has the form $f(x)=Ax+b$, where $A\in O(n)$ and $b\in\mathbb{R}^n$. (We regard $x,b\in\mathbb{R}^n$ as column vectors.)

We write: $f=(A,b)$ for short.

Proof: If $f(\underline{0})=b$, then

$$t_b^{-1}\circ f = t_{-b}\circ f$$

is an isometry fixing $\underline{0}$. Hence

$$t_b^{-1}\circ f = A \in O(n) \text{ by the Lemma,}$$

and $f = t_b\circ A$, ie.

$$f(x) = t_b(Ax) = Ax+b. \quad //$$

Orientation preserving/reversing isometries

Each orthogonal matrix $A \in O(n)$ has determinant ± 1, since

$$1 = \det(I) = \det(A^T A) = \det A^T \det A = (\det A)^2 .$$

If $\det A = +1$, then A represents a "rotation"; if $\det A = -1$, then A represents a reflection composed with a rotation.

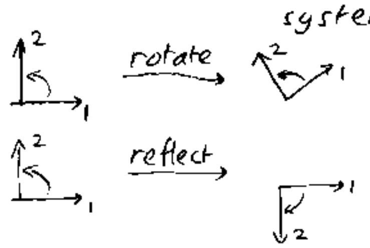
More generally, isometries (A,b) (i.e. $x \mapsto Ax+b$) are called orientation preserving if $\det A = +1$, & orientation reversing if $\det A = -1$.

Idea:

"right handed" coordinate systems $\xrightarrow{\text{orient. preserving}}$ "right handed" coordinate systems

"right handed" coordinate systems $\xrightarrow{\text{orient. reversing}}$ "left handed" coordinate systems

e.g.



(det = +1)

(det = −1)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 180^\circ \text{ rotation around } z\text{-axis} \qquad (\det = +1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \text{reflection in plane } z=0 \qquad (\det = -1)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \text{"inversion in the origin"} \qquad (\det = -1)$$

$$(x,y,z) \mapsto (-x,-y,-z)$$

THE ISOMETRY GROUP OF $\mathbb{E}^n$

Now every composition of isometries is an isometry, and each isometry has an inverse which is an isometry (Check!). Hence the set of all isometries of $\mathbb{E}^n$ forms a group $\mathrm{isom}(\mathbb{E}^n)$ using composition of functions as the operation.

In fact,

$$(A,b)\circ(A',b') : x \mapsto A(A'x+b')+b = AA'x + (Ab'+b),$$

so $(A,b)\circ(A',b') = (AA', Ab'+b)$, $(*)$

for $A, A' \in O(n)$, $b, b' \in \mathbb{R}^n$.

It is also easy to check that:

$$(A,b)^{-1} = (A^{-1}, -A^{-1}b) \quad \text{(Exercise!)}$$

[Remark: We can also represent isometries using $(n+1)\times(n+1)$ matrices.

$$\underset{x \mapsto Ax+b}{(A,b)} \longleftrightarrow \begin{array}{c} \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right) \begin{array}{l} \updownarrow n \\ \updownarrow 1 \end{array} \\ \overset{\longleftrightarrow}{n} \quad \overset{\longleftrightarrow}{1} \end{array} \quad , A \in O(n),\ b \in \mathbb{R}^n$$

row of zeros

Then: $(A,b) : \underset{\in \mathbb{R}^{n+1}}{\begin{pmatrix} x \\ 1 \end{pmatrix}} \mapsto \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax+b \\ 1 \end{pmatrix}$,

$$\left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} A' & b' \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} AA' & Ab'+b \\ \hline 0 & 1 \end{array}\right).$$

It follows that $\mathrm{isom}(\mathbb{E}^n)$ is isomorphic to a subgroup of $GL(n+1,\mathbb{R})$.]

<u>Algebraic structure of $\mathrm{isom}(\mathbb{E}^n)$</u>:

Notice that the set of all translations $T = \{t_a : a \in \mathbb{R}^n\}$ is a subgroup of $\mathrm{isom}(\mathbb{E}^n)$, since compositions & inverses of translations are translations. In fact, T is isomorphic to the group $(\mathbb{R}^n, +)$ with vector addition as operation, since $t_a \cdot t_b = t_{a+b}$, for all $a, b \in \mathbb{R}^n$.

The next result is the key to understanding the structure of the group $\mathrm{isom}(\mathbb{E}^n)$ — it shows how the group is built up from $T \cong (\mathbb{R}^n, +)$ and $O(n)$.

<u>THEOREM</u>: There is a homomorphism

$$\pi : \mathrm{isom}(\mathbb{E}^n) \longrightarrow O(n)$$

defined by taking the "orthogonal part" of each isometry: $\pi((A,b)) = A$.
This is onto, with kernel $T = \{\text{all translations}\}$.
So T is a normal subgroup of $\mathrm{isom}(\mathbb{E}^n)$, and $\mathrm{isom}(\mathbb{E}^n)/T \cong O(n)$.

Further for each $A \in O(n)$, $t_b \in T$:

$$A\, t_b\, A^{-1} = t_{Ab} \;.$$

<u>Proof</u>: π is a homomorphism since

$$\pi\big((A,b)\circ(A',b')\big) = \pi\big((AA', Ab'+b)\big)$$
$$= AA' = \pi((A,b)) \cdot \pi((A',b')),$$

by the composition formula (*).

$\ker \pi$ consists of the isometries (I, b), i.e. translations $x \mapsto Ix + b = x + b = t_b(x)$.
So $T = \ker \pi$ is a normal subgroup.
Clearly π is onto, so

$$O(n) = \mathrm{im}\, \pi \cong \mathrm{isom}(\mathbb{E}^n)/\ker \pi = \mathrm{isom}(\mathbb{E}^n)/T,$$

by the isomorphism theorem.

Finally, for $A \in O(n)$, $t_b \in T$:

$$A\, t_b\, A^{-1} : x \mapsto A^{-1}x \mapsto A^{-1}x + b$$
$$\mapsto A(A^{-1}x + b) = x + Ab,$$

so $A\, t_b\, A^{-1} = t_{Ab}$.

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<u>Next</u>: We want to understand the effect of isometries in more detail!