Recall:

- Isometries of $\mathbb{E}^n$ are distance preserving functions: $\mathbb{R}^n \to \mathbb{R}^n$
- Each isometry has the form:

$(A,b): x \longmapsto Ax+b, \quad A \in O(n), b, x \in \mathbb{R}^n$

- Taking "orthogonal parts" gives a homomorphism $\pi: \mathrm{isom}(\mathbb{E}^n) \longrightarrow O(n)$ with $\ker\pi = \{\text{translations}\}$, $\mathrm{im}\,\pi = O(n)$.
- $x \longmapsto Ax+b$ is
  orientation preserving if $\det A = +1$
  orientation reversing if $\det A = -1$.

Next: Look at the detailed classification of isometries.

Exercise:

(a) $\det: O(n) \to \{\pm 1\}$ is a homomorphism with $\ker = SO(n) =$ "rotations", $\mathrm{im} = \{\pm 1\}$.

$\Rightarrow SO(n)$ is a normal subgroup of index 2 in $O(n)$ with $O(n)/SO(n) \cong \{\pm 1\}$.

(b) The orientation preserving isometries $\mathrm{isom}_+(\mathbb{E}^n)$ form a normal subgroup of index 2 in $\mathrm{isom}(\mathbb{E}^n)$.

[Hint: they form the kernel of the homomorphism $\det \circ \pi: \mathrm{isom}(\mathbb{E}^n) \to \{\pm 1\}$.]

Note also: under composition we have

$\cdot$	o.p.	o.r.
o.p.	o.p	o.r.
o.r.	o.r.	o.p

(1) The group $\mathrm{isom}(\mathbb{E}^n)$ acts on $\mathbb{E}^n$ in a natural way:

$$(A,b)\cdot x = Ax+b, \quad \text{for } x \in \mathbb{R}^n.$$

The orbit of each point p is $\mathbb{R}^n$, since we can translate p to any other point.

The stabilizer of the origin $\underline{0}$ $= \{$all isometries fixing $\underline{0}\}$ is the orthogonal group $O(n)$, by the Lemma.

In general, the stabilizer of p is the conjugate subgroup

$$t_p\, O(n)\, t_p^{-1} = \{t_p A t_p^{-1} : A \in O(n)\},$$

where t_p = translation by p.

Proof: $f \in \mathrm{Stab}(p)$

$\Leftrightarrow \quad f(p) = p$

$\Leftrightarrow \quad f \circ t_p(\underline{0}) = t_p(\underline{0})$

$\Leftrightarrow \quad t_p^{-1} \circ f \circ t_p(\underline{0}) = \underline{0}$

$\Leftrightarrow \quad t_p^{-1} f\, t_p \in \mathrm{Stab}(\underline{0}) = O(n)$.

So: $t_p^{-1} f\, t_p = A \in O(n)$,

and $f = t_p A t_p^{-1}$. //

More explicitly, if $f \in \mathrm{isom}(\mathbb{E}^n)$ fixes p, then

$$f(x) = A(x-p) + p,$$
where $A \in O(n)$.

[Check: $f(p) = A(p-p)+p = 0+p = p$.]

② Isometries of $\mathbb{E}^2$

An orthogonal matrix in $O(2)$ is a 2×2 real matrix with orthonormal columns.

$\therefore$ We have either:

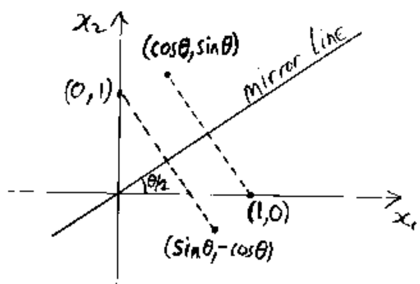
(i) $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with $\det A = +1$,

representing a rotation by θ anticlockwise around the origin.

or (ii) $B_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, with $\det B = -1$,

representing a reflection in the line through the origin at angle $\frac{\theta}{2}$ from the x_1-axis.

Picture:



We also have $B_\theta = A_\theta r$, where $r = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents reflection in the x_1-axis. (Check!)

Thus each isometry of $\mathbb{E}^2$ has the form

$x \mapsto A_\theta x + b$ (orient. pres)

or $x \longmapsto B_\theta x + b = A_\theta r x + b$ (orient. rev)

where $\theta \in \mathbb{R}$ and $b \in \mathbb{R}^2$.

③ Complex notation:

We can think of $\mathbb{E}^2$ as the complex plane $\mathbb{C}$, where

$(x, y) \in \mathbb{R}^2 \longleftrightarrow z = x + iy \in \mathbb{C}$.

Then (check this!)

(i) A_θ = rotation by θ : $z \mapsto e^{i\theta} z$

(ii) r = reflection in real axis:

$z \mapsto \bar{z}$

(iii) t_b = translation by b : $z \mapsto z + b$

$\therefore$ Every isometry of $\mathbb{E}^2$ has the form:

$z \mapsto e^{i\theta} z + b$ (if orient. pres.)

or $z \mapsto e^{i\theta} \bar{z} + b$ (if orient. rev.)

[This only applies in dimension 2!]

THEOREM (Classification of isometries of $\mathbb{E}^2$)

Every isometry of $\mathbb{E}^2$ is either: a rotation, translation, reflection, glide reflection, or the identity.

Note: (1) A glide reflection is a reflection in a line followed by translation parallel to that line.

e.g.

(2) We usually consider the identity as a rotation (by 0) and as a translation (by $\underline{0}$).

Summary (for non-identity isometries):

	Fixed points	No fixed points
orient. preserving	rotation	translation
orient. reversing	reflection	glide reflection

Proof (There are many other approaches!)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry of $\mathbb{E}^2$.

Case 1: If f fixes a point p (i.e. $f(p) = p$), then f is conjugate to an element of $O(2)$. In fact,

$$f = t_p A t_p^{-1} \text{ where } A \in O(2)$$

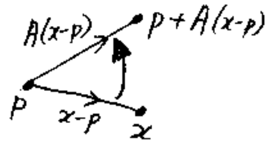
(from our calculation of stabilizers).

Explicitly,

$$f : x \mapsto A(x-p) + p.$$

This means that f is a rotation or reflection fixing p.

Sketch:



Case 2: If $f = (A, b)$ has no fixed points, then the equation

$$f(x) = Ax + b = x,$$

or $\quad (A - I)x = -b$

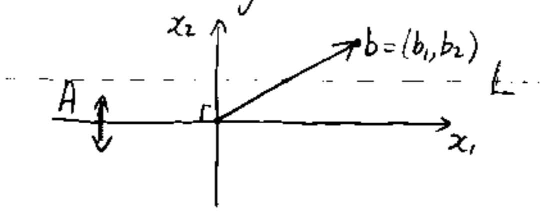
has no solution. Thus $A - I$ is not invertible, so $\det(A - I) = 0$, and $\lambda = 1$ is an eigenvalue of A.

- If f is orientation preserving, then $\det A = +1$ and A is a rotation by 0, i.e. $A = I$. Hence

  $$f(x) = Ix + b = x + b,$$

  and f is a translation.

- If f is orientation reversing, then $\det A = -1$ and A is a reflection. Choose coordinate axes so that the x_1-axis is the mirror line for A, and the x_2-axis is orthogonal to this:



Then

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} x_1 + b_1 \\ -x_2 + b_2 \end{bmatrix},$$

and the line L where $x_2 = \frac{1}{2} b_2$ is mapped to itself by f.

In fact, f is just reflection in L followed by translation by $(b_1, 0)$ parallel to L, since

$$f : (x_1, x_2) \mapsto (x_1, b_2 - x_2) \mapsto (x_1 + b_1, b_2 - x_2).$$

Hence f is a glide reflection. //

(See notes for another approach.)