Classification of isometries of $\mathbb{E}^2$:

For isometries $\neq$ identity:

	Fixed points	No fixed points
orient. preserving	rotation	translation
orient. reversing	reflection	glide reflection

For composition:

	o.p.	o.r.
o.p.	o.p.	o.r.
o.r.	o.r.	o.p.

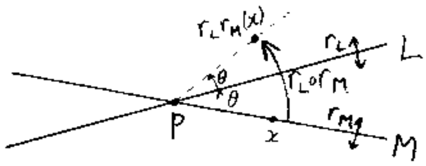
COMBINING ISOMETRIES

The classification of isometries in $\mathbb{E}^2$ gives a lot of information about compositions of isometries.

Example: Let r_L, r_M be reflections in lines L, M. Then $r_L \circ r_M$ is a product of two orientation reversing transformations, so is orientation preserving. Hence $r_L \circ r_M$ is a rotation or translation.

Case 1: If the lines intersect, say $L \cap M = \{p\}$, then p is fixed by r_L and r_M, so is fixed by $r_L \circ r_M$.

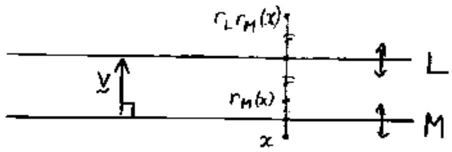
Thus $r_L \circ r_M$ is a rotation about p.



In fact, $r_L \circ r_M$ is a rotation by twice the angle from M to L.

(e.g. Look at $r_L r_M(x)$ for $x \in M$ as above.)

Case 2: If the lines L, M are parallel, then $r_L \circ r_M$ is a translation by $2\underline{v}$, where $\underline{v}$ = vector from M to L orthogonal to L and M.

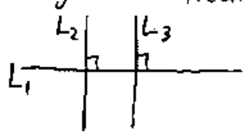


Exercise: Check this.
(e.g. choose coordinates so M, L are the lines $x_2 = a$, $x_2 = b$)

From this example we see that the isometry group $\mathrm{Isom}(\mathbb{E}^2)$ is generated by reflections. In fact,

Theorem: Every isometry of $\mathbb{E}^2$ is a product of at most 3 reflections.

(1 for a reflection,
2 for a rotation or translation,
3 for a glide reflection:



$r_{L_3} r_{L_2} r_{L_1}$ = glide with axis L_1.)

(In $\mathbb{E}^n$, each isometry is a product of at most $n+1$ reflections in hyperplanes, i.e. $(n-1)$-dimensional planes.)

Other products of isometries can be understood (i) algebraically, or (ii) geometrically (by writing isometries as products of reflections).

<u>Example</u>: Let R_1 be a rotation by α (anticlockwise) around a, and R_2 be rotation by β about b.

- If $a=b$, $R_2 \circ R_1$ is rotation by $\alpha+\beta$ around $a=b$.

- If $a \neq b$, $R_2 \circ R_1$ is an orientation preserving isometry, so is either a rotation or translation.

(i) <u>Algebraic approach</u>: $(A,b): x \mapsto Ax+b$

$$R_2 \circ R_1 = (A_\beta, \underline{v}_2) \circ (A_\alpha, \underline{v}_1) = (A_\beta A_\alpha, ?)$$

has orthogonal part $A_{\alpha+\beta}$ = rotation by $\alpha+\beta$. So

(a) if $\alpha+\beta \neq$ multiple of 2π, $R_2 \circ R_1$ is a rotation by $\alpha+\beta$ about some point.

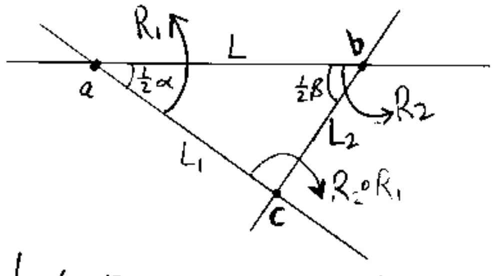
(b) if $\alpha+\beta =$ multiple of 2π, then $R_2 \circ R_1$ is a translation.

(ii) <u>Geometric approach</u>

Given the rotation centres a, b, let L = line through a, b

L_1 = line through a at angle $-\frac{1}{2}\alpha$ from L,

L_2 = line through b at angle $\frac{1}{2}\beta$ from L



Let r, r_1, r_2 = reflections in L, L_1, L_2 respectively.

Then:

R_1 = rotation by α about $a = r \circ r_1$

R_2 = rotation by β about $b = r_2 \circ r$

$\therefore\ R_2 \circ R_1 = r_2 \circ r \circ r \circ r_1$

$\qquad = r_2 \circ r_1$ since $r \circ r$ = identity.

$\therefore\ R_2 \circ R_1$ = <u>rotation</u> around the intersection point <u>c of L_1, L_2</u>, by <u>twice the angle from L_1 to L_2</u>.

$\therefore$ rotation angle is

$2(\pi - \frac{1}{2}\alpha - \frac{1}{2}\beta) = 2\pi - \alpha - \beta$ clockwise, i.e. $\alpha+\beta$ anticlockwise.

[If $\alpha+\beta$ = multiple of 2π, then L_1, L_2 are <u>parallel</u>, and $R_2 \circ R_1$ = translation.]

This approach extends to rotations in 3-dimensions (write them as products of 180° rotations), and to other geometries, e.g. "spherical geometry" = geometry on the sphere, and "hyperbolic geometry".

Conjugacy classes of isometries

If $f, g \in \mathrm{isom}(\mathbb{E}^2)$, then the conjugate gfg^{-1} represents the "same" isometry as f after "changing coordinates by g":

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & f(x) \arrow[d, "g"] \\ g(x) \arrow[r, "gfg^{-1}"] & g(f(x)) \end{tikzcd}$$

Then it's easy to check:

(a) p is a fixed point of f
$\iff g(p)$ is a fixed point of gfg^{-1}.

(b) f is reflection in line L
$\Rightarrow gfg^{-1}$ is reflection in line $g(L)$

(c) f is a rotation by θ about p
$\Rightarrow gfg^{-1}$ is a rotation by $\pm\theta$ about $g(p)$
($+\theta$ if g is orient. preserving,
$-\theta$ if g is orient. reversing).
[<u>Hint</u>: write f as a product of reflections & use (b).]

(d) f is a translation by $\underset{\sim}{b}$,
$\Rightarrow gfg^{-1}$ is a translation by $\pi(g)(\underline{b})$
[$\pi(g)$ = orthogonal part of g.]

(e) f is a glide reflection with axis L
$\Rightarrow gfg^{-1}$ is a glide reflection with axis $g(L)$.
[and if f translates by $\underline{b}$ along L,
then gfg^{-1} " " $\pi(g)\underline{b}$ along $g(L)$.]