Symmetries of Patterns in $\mathbb{E}^2$

Next we will study the possible symmetry groups of objects & patterns in the plane, i.e. subgroups of $\mathrm{isom}(\mathbb{E}^2)$ taking an object or pattern to itself.

Classification of finite symmetry groups:

Theorem (Leonardo da Vinci?)

If G is a finite subgroup of $\mathrm{isom}(\mathbb{E}^2)$ then G is a cyclic group C_n (rotations round a point by multiples of $\frac{2\pi}{n}$), or a dihedral group D_n (symmetries of a regular n-gon).

[redacted]

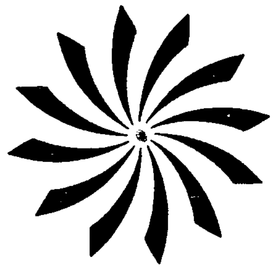
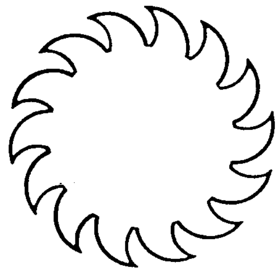


Figure 11.11 Multiple rotations.



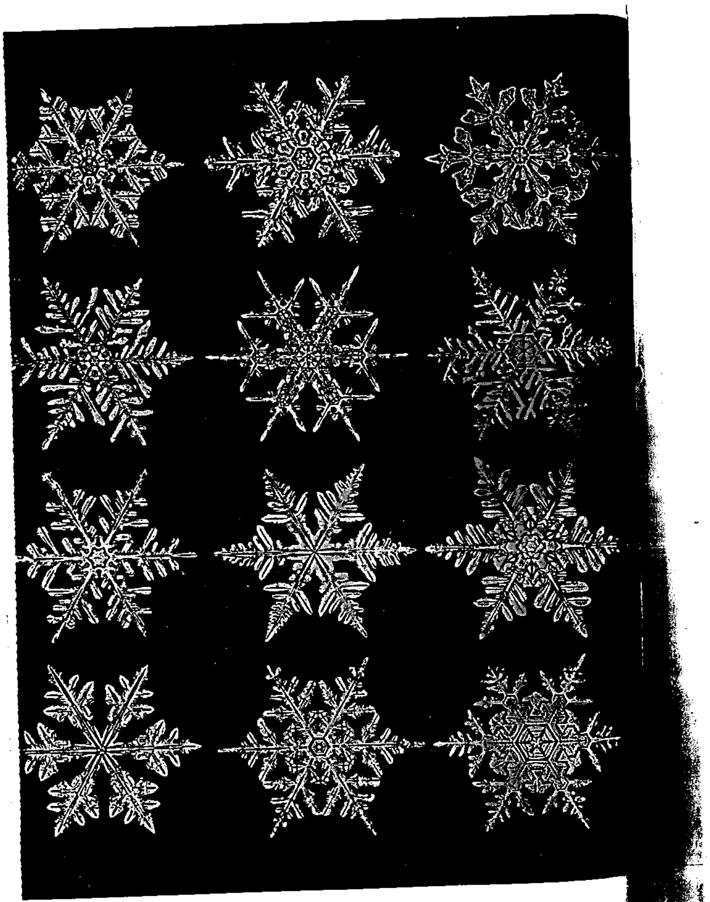


FIG. 38

The first step is the following

FIXED POINT THEOREM:

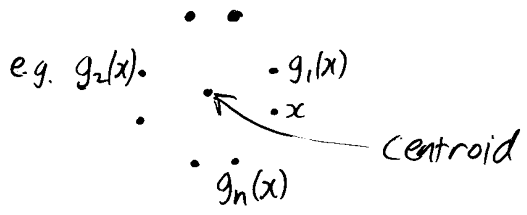
Let G be a finite subgroup of $\mathrm{isom}(\mathbb{E}^2)$. Then there is a point p in $\mathbb{E}^2$ which is fixed by G, i.e. $g(p)=p$ for all $g \in G$.

Proof: Pick any $x \in \mathbb{E}^2$ and we consider the "centre of gravity" or "centroid" p of its orbit $G \cdot x$.

Explicitly, if $G = \{g_1, \dots, g_n\}$ then

$$p = \frac{1}{|G|} \sum_{g \in G} g(x) = \frac{1}{n} \sum_{i=1}^{n} g_i(x) .$$

(i.e. average position of the points $g_i(x)$ in the orbit of x.)



<u>Claim</u>: p is <u>fixed</u> by each $g \in G$

(1) For $g \in G$,

$$g(y) = Ay + b, \text{ with } A \in O(2),\ b \in \mathbb{R}^2.$$

$$\begin{aligned}
\therefore\ & g\left(\tfrac{1}{n}(y_1 + \dots + y_n)\right) \\
&= A\left(\tfrac{1}{n}(y_1 + \dots + y_n)\right) + b \\
&= \tfrac{1}{n}(Ay_1 + \dots + Ay_n) + b \quad \text{(linearity of } A\text{)} \\
&= \tfrac{1}{n}\left[(Ay_1 + b) + \dots + (Ay_n + b)\right] \\
&= \tfrac{1}{n}\left(g(y_1) + \dots + g(y_n)\right)
\end{aligned}$$

So g takes the centroid of $(y_1, \dots, y_n)$ to the centroid of $(g(y_1), \dots, g(y_n))$.

(2) $gG = \{gg_1, \dots, gg_n\} = G$, since left multiplication by g permutes the elements of G.

Hence, for each $g \in G$.

$$\begin{aligned}
g(p) &\overset{(1)}{=} \text{centroid of } gg_1(x), \dots, gg_n(x) \\
&\overset{(2)}{=} \text{centroid of } g_1(x), \dots, g_n(x) \\
&= p.
\end{aligned}$$

This proves the fixed point theorem. //

<u>Remark</u>: The same argument shows that every finite subgroup of $\operatorname{isom}(\mathbb{E}^n)$ fixes a point in $\mathbb{E}^n$.

Now we know that G is a finite group of rotations and reflections fixing the point p. (G is a subgroup of the stabilizer of p $= t_p O(2) t_p^{-1}$ = a conjugate of $O(2)$.)

<u>Case 1</u>: Assume G contains only <u>rotations</u>. If $G \neq \{e\}$, choose a rotation $r \in G$ by the smallest positive angle, say θ.

<u>Claim</u>: G is generated by r, and $\theta = \frac{2\pi}{n}$ for some integer n. So $G \cong C_n$.

<u>Proof</u>: Suppose G contains a rotation s by angle α. Then we can write

$$\alpha = k\theta + \beta, \text{ where } k \in \mathbb{Z}$$
and $0 \leq \beta < \theta$.

Then $sr^{-k} \in G$ and is a rotation by $\alpha - k\theta = \beta$. Hence $\beta = 0$, otherwise θ would not be the smallest positive rotation angle. So $\alpha = k\theta$, and $s = r^k$. Thus G is a cyclic group generated by r.

Further, let $n\theta$ be the smallest multiple of θ which is $\geq 2\pi$. Then

$$2\pi \leq n\theta < 2\pi + \theta.$$

Since θ is the smallest positive rotation angle, this implies $n\theta = 2\pi$. (Otherwise $0 < 2\pi - n\theta < \theta$, and $2\pi - n\theta$ would be a smaller rotation angle.) Thus, $\theta = \frac{2\pi}{n}$ for some integer n.

Case 2: Assume G contains a reflection m. Let H be the subgroup of G consisting of rotations. Than H is a cyclic group generated by a rotation r through an angle $\frac{2\pi}{n}$, for some integer $n \geq 1$.

Now r, m generate a dihedral group $D_n = \langle r, m : r^n = 1, m^2 = 1, mrm = r^{-1} \rangle$, which is a subgroup of G.

Claim: $G = D_n$.

Proof: If $g \in G$ is a rotation, then $g = r^k$ for some k, so $g \in D_n$.
If $g \in G$ is a reflection, then gm is a rotation in G, so $gm = r^k$ for some integer k.

Hence $g = r^k m$ (since $m^{-1} = m$), and $g \in D_n$.

This shows that $G \subseteq D_n$, hence $G = D_n$.

Thus every finite subgroup of $\mathrm{isom}(\mathbb{E}^2)$ is isomorphic to C_n or D_n. //

In 3-dimensions, every finite group of isometries is conjugate to a subgroup of $O(3)$.

The finite subgroups of $SO(3)$ (i.e. the finite groups of rotations), are: C_n, D_n, T, O, I

where C_n = cyclic group
D_n = dihedral group

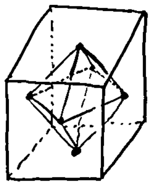
T = "tetrahedral group"
O = "octahedral group"
I = "icosahedral group"

i.e. rotational symmetries of a regular tetrahedron,
octahedron (or cube),
icosahedron (or dodecahedron).

Note: ① $|T| = 12$
$|O| = 24$
$|I| = 60$

② The cube and octahedron are "dual" and have isomorphic symmetry groups.

(centres of faces in cube $\longleftrightarrow$ vertices in octahedron and vice versa).



Similarly the dodecahedron and icosahedron are dual, so have isomorphic symmetry groups.

③ T, O, I correspond to nice permutation groups.

DEFINITION: A permutation in S_n is called an even permutation if it can be written as the product of an even number of 2-cycles.

e.g. $(1) = (12)(12)$, $(123) = (13)(12)$, $(12)(34)$ are even

An *odd* permutation can be written as the product of an *odd* number of 2-cycles.

e.g. (12), $(1234) = (14)(13)(12)$, etc.

Or: given $\sigma \in S_n$ form the matrix $M(\sigma)$ with columns: $e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}$

[e_i = ith unit vector in $\mathbb{R}^n$].

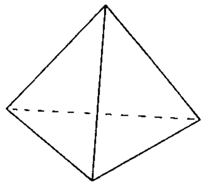
Then σ is *even* if $\det M(\sigma) = +1$

σ is *odd* if $\det M(\sigma) = -1$.

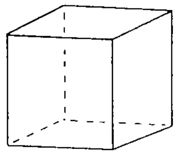
It's not hard to check that the even permutations form a subgroup of S_n, called the *alternating group A_n*, with $|A_n| = \frac{1}{2}(n!)$.

In fact, $T \cong A_4$, $O \cong S_4$, $I \cong A_5$! //

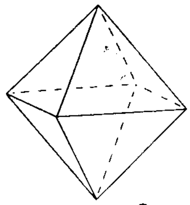
The regular polyhedra (Platonic solids)



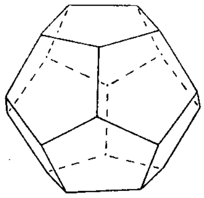
Tetrahedron



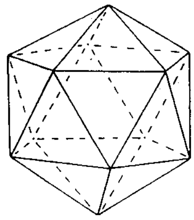
Cube



Octahedron



Dodecahedron



Icosahedron