Symmetries of Patterns in $E^2$

Next we will study the possible symmetry groups of objects & patterns in the plane, i.e. subgroups of $\text{isom}(E^2)$ taking an object or pattern to itself.

Classification of finite symmetry groups:

*Theorem* (Leonardo da Vinci?)

If $G$ is a finite subgroup of $\text{isom}(E^2)$ then $G$ is a cyclic group $C_n$ (rotations round a point by multiples of $\frac{2\pi}{n}$), or a dihedral group $D_n$ (symmetries of a regular $n$-gon).

---

The first step is the following

**Fixed Point Theorem:**

Let $G$ be a finite subgroup of $\text{isom}(E^2)$. Then there is a point $p$ in $E^2$ which is fixed by $G$, i.e. $g(p) = p$ for all $g \in G$.

*Proof:* Pick any $x \in E^2$ and we consider the "centre of gravity" or "centroid" $p$ of its orbit $G \cdot x$.

Explicitly, if $G = \{g_1, \ldots, g_n\}$ then

$$p = \frac{1}{|G|} \sum_{g \in G} g(x) = \frac{1}{n} \sum_{i=1}^{n} g_i(x).$$

(i.e. average position of the points $g_i(x)$ in the orbit of $x$.)
Claim: p is fixed by each \( g \in G \)

(1) For \( g \in G \),
\[
g(\mathbf{y}) = Ay + b, \text{ with } A \in O(2), b \in \mathbb{R}^2.
\]
\[
ineq{g}{\left(\frac{1}{n}(y_1+\ldots+y_n)\right)}
= \frac{1}{n}(Ay_1+\ldots+Ay_n) + b \quad \text{(linearity of } A)\]
= \frac{1}{n}[(Ay_1+b) + \ldots + (Ay_n+b)]
= \frac{1}{n}(g(y_1) + \ldots + g(y_n))
\]
So \( g \) takes the centroid of \((y_1, \ldots, y_n)\)
to the centroid of \((g(y_1), \ldots, g(y_n))\).

(2) \( G = \{g_1, \ldots, g_n\} = G \), since left multiplication by \( g \) permutes the elements of \( G \).

Hence, for each \( g \in G \),
\[
g(p) = \frac{1}{n}(\text{centroid of } g_1(x), \ldots, g_n(x))
= p.
\]

This proves the fixed point theorem. \( \Box \)

Remark: The same argument shows that every finite subgroup of \( \text{Isom}(\mathbb{E}^n) \) fixes a point in \( \mathbb{E}^n \).

Now we know that \( G \) is a finite group of rotations and reflections fixing the point \( p \). (\( G \) is a subgroup of the stabilizer of \( p \)
\( = tpO(2)tp^{-1} = \text{a conjugate of } O(2). \))

Case 1: Assume \( G \) contains only rotations.

If \( G \neq \{e\} \), choose a rotation \( r \in G \)
by the smallest positive angle, say \( \theta \).

Claim: \( G \) is generated by \( r \), and \( \theta = \frac{2\pi}{n} \)
for some integer \( n \).

So \( G = C_n \).

Proof: Suppose \( G \) contains a rotation \( s \)
by angle \( \alpha \). Then we can write
\[
\alpha = k\theta + \beta, \text{ where } k \in \mathbb{Z}
\]
and \( 0 \leq \beta < \theta \).

Then \( Sr^k \in G \) and is a rotation
by \( \alpha = k\theta + \beta \). Hence \( \beta = 0 \),
otherwise \( \theta \) would not be the smallest positive rotation angle. So \( \alpha = k\theta \),
and \( s = r^k \). Thus \( G \) is a cyclic group generated by \( r \).

Further, let \( n\theta \) be the smallest multiple of \( \theta \) which is \( > 2\pi \).
Then
\[
2\pi \leq n\theta < 2\pi + \theta.
\]

Since \( \theta \) is the smallest positive rotation angle, this implies \( n\theta = 2\pi \).
(Otherwise \( 0 < 2\pi - n\theta < \theta \), and \( 2\pi - n\theta \) would be a smaller rotation angle)
Thus, \( \theta = \frac{2\pi}{n} \) for some integer \( n \).
Case 2: Assume \( G \) contains a reflection \( m \).
Let \( H \) be the subgroup of \( G \) consisting of rotations. Then \( H \) is a cyclic group generated by a rotation \( r \) through an angle \( \frac{2\pi}{n} \), for some integer \( n \geq 1 \).

Now \( r, m \) generate a dihedral group \( D_n = \langle r, m : r^n = 1, m^2 = 1, mrm^{-1} = r^{-1} \rangle \).
which is a subgroup of \( G \).

Claim: \( G = D_n \).

Proof: If \( g \in G \) is a rotation, then \( g = r^k \) for some \( k \), so \( g \in D_n \).

If \( g \in G \) is a reflection, then \( gm \) is a rotation in \( G \), so \( gm = r^k \) for some integer \( k \).

Hence \( g = r^k m \) (since \( m^2 = m \)), and \( g \in D_n \).

This shows that \( G \subseteq D_n \), hence \( G = D_n \).

Thus every finite subgroup of \( \text{isom}(\mathbb{R}^2) \) is isomorphic to \( C_n \) or \( D_n \).

In 3-dimensions, every finite group of isometries is conjugate to a subgroup of \( O(3) \).

The finite subgroups of \( O(3) \) (i.e., the finite groups of rotations) are: \( C_n, D_n, T, O, I \).

where \( C_n = \) cyclic group
\( D_n = \) dihedral group

\( T = \) "tetrahedral group"
\( O = \) "octahedral group"
\( I = \) "icosahedral group"

i.e. rotational symmetries of a regular tetrahedron,
- octahedron (or cube),
- icosahedron (or dodecahedron).

\[ |T| = 12 \]
\[ |O| = 24 \]
\[ |I| = 60 \]

Note: 0

2 The cube and octahedron are "dual" and have isomorphic symmetry groups.

3 \( T, O, I \) correspond to nice permutation groups.

**Definition:** A permutation in \( S_n \) is called an even permutation if it can be written as the product of an even number of 2-cycles.

E.g. \( (1) = (12)(12), \ (123) = (13)(12), \ (12)(34) \) are even
An odd permutation can be written as the product of an odd number of 2-cycles: e.g. \((12), (1234) = (14)(13)(12), \) etc.

Or: given \(\sigma \in S_n\) form the matrix \(M(\sigma)\) with columns: \(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}\)

\[ e_i : \text{ith unit vector in } \mathbb{R}^n \]

Then \(\sigma\) is even if \(\det M(\sigma) = +1\)

\(\sigma\) is odd if \(\det M(\sigma) = -1\).

It's not hard to check that the even permutations form a subgroup of \(S_n\), called the alternating group \(A_n\), with \(|A_n| = \frac{1}{2}n!\).

In fact, \(T \cong A_4\), \(O \cong S_4\), \(I \cong A_5\)! //