**DISCRETE GROUPS IN $E^2$**

We now want to study the symmetries of repeating patterns in the plane $E^2$.

**References:**
- Notes §2.9
- M. Artin, Algebra, chapter 5

We will look at discrete symmetry groups, i.e., exclude:

1. patterns with translations by arbitrarily short distances

   e.g. line

2. patterns with rotations by arbitrarily small angles

   e.g. Circle

**Definition:** A subgroup $G$ of $\text{isom}(E^2)$ is discrete if there is $\varepsilon > 0$ s.t.

- (i) each non-zero translation in $G$ has translation distance $\geq \varepsilon$, and
- (ii) each non-trivial rotation in $G$ has rotational angle $\geq \varepsilon$.

(We don't need to put any condition on reflections or glide reflections.)

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See handout and the end of the printed notes for some sample patterns with discrete symmetry groups!

Given any discrete subgroup $G$ of $\text{isom}(E^2)$ we can built up patterns with $G$ as symmetry group as follows:

Start with a figure (e.g. $R$) with no symmetries except the identity. Then apply all the elements of $G$ to the figure.

(The computer program Kali lets you build up beautiful patterns in this way—as in a Kaleidoscope.)

Recall that each isometry of $E^2$ is built up from an orthogonal part and a translational part:

$(a, b) : x \mapsto Ax + b$, $A \in O(2)$, $b \in \mathbb{R}^2$

Further there is a homomorphism

$\pi : \text{isom}(E^2) \to O(2)$

with $\ker \pi = T = \{ \text{all translations} t : \varepsilon(\mathbb{R}^2, +) \}$

We now analyse a discrete group $G \subset \text{isom}(E^2)$ by studying

(i) the translational subgroup

$G \cap T = \text{subgroup of } T$

(ii) the “point group”

$G = \pi(G) = \text{subgroup of } O(2)$. 
e.g. For $G = \text{symmetries of hexagonal tiling}$:

- **Translational subgroup** $G \times T$:
  generated by translations $T_1, T_2$.

- **Point group** $\overline{G} \cong \Pi(G) \cong D_6$
  generated by:
  $$\begin{bmatrix}
  \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\
  \sin \frac{\pi}{3} & \cos \frac{\pi}{3}
  \end{bmatrix},
  \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
  (rotation by $\frac{\pi}{3}$) (reflection)

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**CAUTION:** The point group $\Pi(G)$ need not be a subgroup of $G$.

- **Example:** $G = \text{generated by a glide reflection}$:
  $$\ldots R B R B R \ldots$$

- **$G \times T$** generated by one translation $T$.

- **Point group** $\overline{G} \cong \Pi(G) \cong D_6$
  generated by one reflection

  But $G$ contains no element of order 2 (only translations & glides).

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**Classification of discrete subgroups of $\text{isom}(\mathbb{R}^2)$**

1. **Translation subgroups**:
   $$G \times T = \{\text{all translations in } G\}$$
   corresponds to a subgroup of $(\mathbb{R}^2, +)$:
   $$LG = \{a \in \mathbb{R}^2 : t \in G \}$$
   *translation by $a$.

   **Note:** $G$ discrete $\Rightarrow$ $LG$ is also discrete:
   *there is $\varepsilon > 0$ such that $LG$
   contains no vector of length $< \varepsilon$, except the zero vector.

   **This means:** that distinct vectors in $LG$
   are separated by at least $\varepsilon$. 

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For hexagonal tiling with shading:

- **Translational subgroup**
  is the same (generated by $T_1, T_2$).

- **Point group** $\overline{G} \cong D_6$
  generated by:
  $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
Proposition 1: Every discrete subgroup \( L \) of \( \mathbb{R}^2 \) is either
(i) \( L = \{0\} \)
(ii) \( L \cong \mathbb{Z} \), generated by a non-vector \( a \):
\[
L = \{ na : n \in \mathbb{Z} \}
\]
or (iii) \( L \cong \mathbb{Z} \times \mathbb{Z} \), generated by two linearly independent vectors \( a, b \):
\[
L = \{ na + mb : n, m \in \mathbb{Z} \}
\]
(Then \( L \) is called a lattice in \( \mathbb{R}^2 \).)

Proof: See Artin, Prop 4.5.

In (ii), take \( a = a \) a non-zero vector of shortest length in \( L \).

In (iii), can take \( a = a \) shortest non-zero vector in \( L \), \( b = b \) shortest vector not a multiple of \( a \).

The point group \( \overline{G} = \pi(G) \subseteq O(2) \).

From the classification of isometries, \( g = (A, b) : x \mapsto Ax + b \) is a non-trivial rotation about some point \( \iff \pi(g) = A \subseteq O(2) \) is a non-trivial rotation.

So: if \( G \) is discrete, so is \( \overline{G} = \pi(G) \subseteq O(2) \).

Lemma: A discrete subgroup of \( O(2) \) is finite.

Proof: Exercise (Idea: rotation angles are separated by at least \( \varepsilon > 0 \) \( \Rightarrow \) there are only finitely many rotations.)

Corollary: The point group \( \overline{G} = \pi(G) \) is cyclic or dihedral.

(From the result of last lecture.) Combining these facts about the translation group \( L_G \) & point group \( \overline{G} \) gives a broad classification of discrete groups \( G \) into 3 classes:

(i) If \( L_G = \{0\} \), then \( \pi : G \to \pi(G) = \overline{G} \) has trivial kernel \( \ker \pi = L_G = \{0\} \).

So \( G \cong \overline{G} \) is finite; hence cyclic or dihedral.

(This gives infinitely many groups!)
(i) If $L_G \cong \mathbb{Z}$, then we obtain symmetries of "frieze patterns" or "strip patterns".
It turns out that there are 7 possible symmetry groups for these!

(iii) If $L_G \cong \mathbb{Z} \times \mathbb{Z}$, we obtain the "wallpaper groups."
It turns out that there are exactly 17 of these groups!

**Question:** Why are there only finitely many possibilities in case (ii), (iii)?