Discrete Groups in $\mathbb{E}^2$

We now want to study the symmetries of repeating patterns in the plane $\mathbb{E}^2$.

References:

Notes §2·9

M. Artin, Algebra, chapter 5

M. Armstrong, Groups & Symmetry, chap 25/26.

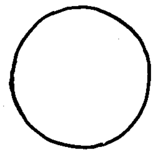
We will look at discrete symmetry groups, i.e. exclude:

(1) patterns with translations by arbitrarily short distances

e.g. line

(2) patterns with rotations by arbitrarily small angles

e.g. circle



DEFINITION: A subgroup G of $\mathrm{isom}(\mathbb{E}^2)$ is discrete if there is $\varepsilon > 0$ s.t.

(i) each non-zero translation in G has translation distance $\geq \varepsilon$, and

(ii) each non-trivial rotation in G has rotational angle $\geq \varepsilon$.

(We don't need to put any condition on reflections & glide reflections.)

See handout and the end of the printed notes for some sample patterns with discrete symmetry groups!

Given any discrete subgroup G of $\mathrm{isom}(\mathbb{E}^2)$ we can built up patterns with G as symmetry group as follows:

Start with a figure (e.g. R) with no symmetries except the identity. Then apply all the elements of G to the figure.

(The computer program Kali lets you build up beautiful patterns in this way – as in a kaleidoscope.)

Recall that each isometry of $\mathbb{E}^2$ is built up from an orthogonal part and a translational part:

$(A,b): x \mapsto Ax + b, \quad A \in O(2),\ b \in \mathbb{R}^2.$

(*) Further there is a homomorphism

$$\pi : \mathrm{isom}(\mathbb{E}^2) \to O(2)$$

with $\ker \pi = T = \{\text{all translations}\} \cong (\mathbb{R}^2, +)$.

We now analyse a discrete group $G \subset \mathrm{isom}(\mathbb{E}^2)$ by studying

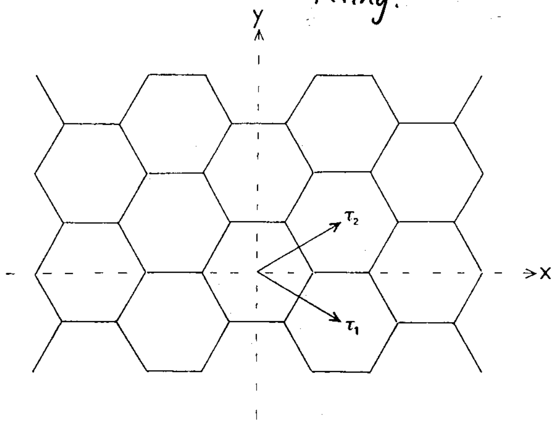
(i) the translational subgroup

$G \cap T$ = subgroup of T

(ii) the "point group"

$\bar{G} = \pi(G)$ = subgroup of $O(2)$.

e.g. For G = symmetries of hexagonal tiling:

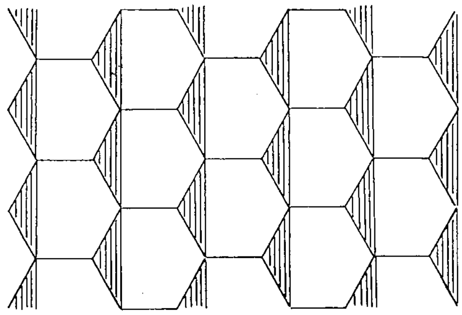


- translational subgroup $G \cap T$: generated by translations τ_1, τ_2.
- point group $\bar{G} = \pi(G) \cong D_6$ generated by

$$\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(rotation by $\frac{\pi}{3}$) (reflection)

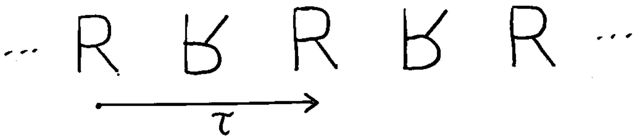
For hexagonal tiling with shading:



- translational subgroup is the same (generated by τ_1, τ_2)
- point group $\bar{G} \cong D_1$ generated by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

CAUTION: The point group $\pi(G)$ need not be a subgroup of G.

e.g. G generated by a glide reflection:



- $G \cap T$ generated by one translation τ.
- point group $\bar{G} = \pi(G) \cong D_1$ generated by one reflection

But G contains no element of order 2 (only translations & glides).

CLASSIFICATION OF DISCRETE SUBGROUPS OF $\mathrm{Isom}(\mathbb{E}^2)$ (Outline!)

1. Translation subgroups:

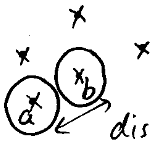
$G \cap T = \{\text{all translations in } G\}$ corresponds to a subgroup of $(\mathbb{R}^2, +)$:

$$L_G = \{a \in \mathbb{R}^2 : t_a \in G\}$$

t_a ← translation by a.

Note: G discrete $\Rightarrow L_G$ is also discrete: i.e. there is $\varepsilon > 0$ such that L_G contains no vector of length $< \varepsilon$, except the zero vector.

$\boxed{\text{This means that distinct vectors in } L_G \text{ are separated by at least } \varepsilon.}$



distance: $d(a,b) = \|a-b\| \geq \varepsilon$, since $a-b \in L$.

<u>Proposition 1</u>: Every discrete subgroup L of $\mathbb{R}^2$ is either

(i) $L = \{\underset{\sim}{0}\}$

(ii) $L \cong \mathbb{Z}$, generated by a non-vector $\underset{\sim}{a}$:

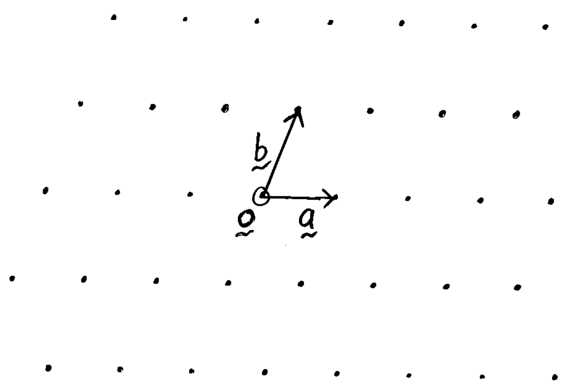
$$L = \{n\underset{\sim}{a} : n \in \mathbb{Z}\}$$

or (iii) $L \cong \mathbb{Z} \times \mathbb{Z}$, generated by two linearly independent vectors $\underset{\sim}{a}, \underset{\sim}{b}$:

$$L = \{n\underset{\sim}{a} + m\underset{\sim}{b} : n, m \in \mathbb{Z}\}$$

(Then L is called a <u>lattice</u> in $\mathbb{R}^2$.)

<u>Picture of L in case (iii):</u>



<u>Proof</u>: See Artin, Prop 4.5.

In (ii), take $\underset{\sim}{a}$ = a non-zero vector of shortest length in L.

In (iii), can take

$\underset{\sim}{a}$ = shortest non-zero vector in L,

$\underset{\sim}{b}$ = shortest vector not a multiple of $\underset{\sim}{a}$

② <u>The point group</u> $\bar{G} = \pi(G) \subseteq O(2)$.

From the classification of isometries,

$g = (A,b) : x \longmapsto Ax + b$

is a non-trivial rotation about some point $\Longleftrightarrow \pi(g) = A \in O(2)$ is a non-trivial rotation.

So: if G is discrete, so is $\bar{G} = \pi(G) \subseteq O(2)$.

<u>Lemma</u>: A discrete subgroup of $O(2)$ is <u>finite</u>.

<u>Pf</u>: <u>Exercise</u> (<u>Idea</u>: rotation angles are separated by at least $\varepsilon > 0$ $\Rightarrow$ there are only finitely many rotations.)

<u>Corollary</u>: The point group $\bar{G} = \pi(G)$ is cyclic or dihedral.

(From the <u>result of last</u> lecture).

Combining these facts about the translation group L_G & point group $\bar{G}$ gives a broad classification of discrete groups G into 3 classes:

(i) If $L_G = \{0\}$, then

$$\pi : G \to \pi(G) = \bar{G}$$

has trivial kernel ($\ker \pi \cong L_G = \{0\}$).

So $G \cong \bar{G}$ is finite; hence cyclic or dihedral.

(This gives infinitely many groups!)

(ii) If $L_G \cong \mathbb{Z}$, then we obtain symmetries of "frieze patterns" or "strip patterns".
It turns out that there are 7 possible symmetry groups for these!

(iii) If $L_G \cong \mathbb{Z} \times \mathbb{Z}$, we obtain the "wallpaper groups".
It turns out that there are exactly 17 of these groups!

QUESTION: Why are there only finitely many possibilities in case (ii), (iii)??