WALLPAPER GROUPS

(2-dimensional crystallographic groups)

These are discrete subgroups G of $\mathrm{isom}(\mathbb{E}^2)$ with <u>translational subgroup</u> $G \cap T \cong \mathbb{Z}^2$ generated by translations by 2 linearly independent vectors, say $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^2$.

We write

$$L_G = \{ n\underset{\sim}{a} + m\underset{\sim}{b} : n, m \in \mathbb{Z} \}$$

for the corresponding <u>"lattice"</u> in $\mathbb{R}^2$, and $\bar{G} = \pi(G)$ for the <u>point group</u> of G consisting of orthogonal parts of all elements in G.

($\bar{G}$ = finite subgroup of $O(2)$.)

The first observation is:

PROPOSITION: Let G be a discrete subgroup of $\mathrm{isom}(\mathbb{E}^2)$. Then the point group $\bar{G}$ takes the lattice L_G to itself, i.e.

$$\bar{g} \in \bar{G},\ a \in L_G \Rightarrow \bar{g}(a) \in L_G.$$

<u>Note</u>: ① This means that $\bar{G}$ <u>acts as a group of symmetries</u> of the lattice L_G.

② The whole group G need <u>not</u> take L_G to itself. (e.g. take $G \cong \mathbb{Z}$ generated by a glide reflection

<u>Proof</u>: Let $\bar{g} \in \bar{G}$ and $a \in L_G$. This means that $t_a \in G \cap T$, and there is $g \in G$ such that

$$g(x) = \bar{g}x + b, \text{ or } g = t_b \circ \bar{g}$$

where $\bar{g} \in O(2)$ and $b \in \mathbb{R}^2$.

$$\text{Now } g t_a g^{-1} = t_b \underbrace{\bar{g} t_a \bar{g}^{-1}} t_b^{-1}$$
$$= t_b t_{\bar{g}(a)} t_b^{-1}$$

(by a previous calculation)

$$= t_{\bar{g}(a)},$$

since translations commute.

But $g t_a g^{-1} \in G$ and is a <u>translation</u>, so $t_{\bar{g}(a)} \in G \cap T$. Hence $\bar{g}(a) \in L_G$. //

We can now deduce that there are only <u>finitely many</u> possibilities for the point group $\bar{G}$.

PROPOSITION ("The crystallographic restriction.")

Let $H \subseteq O(2)$ be a finite group of symmetries of a lattice L. Then

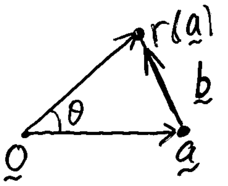
(a) Every rotation in H has order 1, 2, 3, 4 or 6.

(b) H is one of the groups C_n or D_n where $n = 1, 2, 3, 4$ or 6.

[In particular rotational symmetries of order 5 are not possible in a wall paper pattern.]

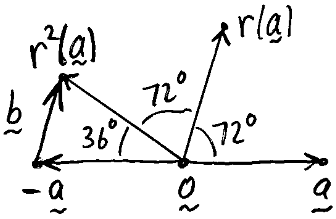
Proof: (a) Let $\mathbf{a}$ be a non-zero vector in L of minimal length. Let $r \in H$ be a rotation by the smallest possible non-zero angle, $\theta = \frac{2\pi}{n}$.

Now $r(\mathbf{a}) \in L$ since $H(L) \subseteq L$ and $\mathbf{a} \in L$; hence $\mathbf{b} = r(\mathbf{a}) - \mathbf{a} \in L$.



But if $n \geq 7$, then $\theta < \frac{2\pi}{6}$ (i.e. 60°), so $|\mathbf{b}| < |\mathbf{a}|$, by simple geometry – contradicting the choice of $\mathbf{a}$ of minimal length.

If $n = 5$ we have:



Then $r^2(\mathbf{a}), -\mathbf{a} \in L$; hence $\mathbf{b} = r^2(\mathbf{a}) + \mathbf{a} \in L$.

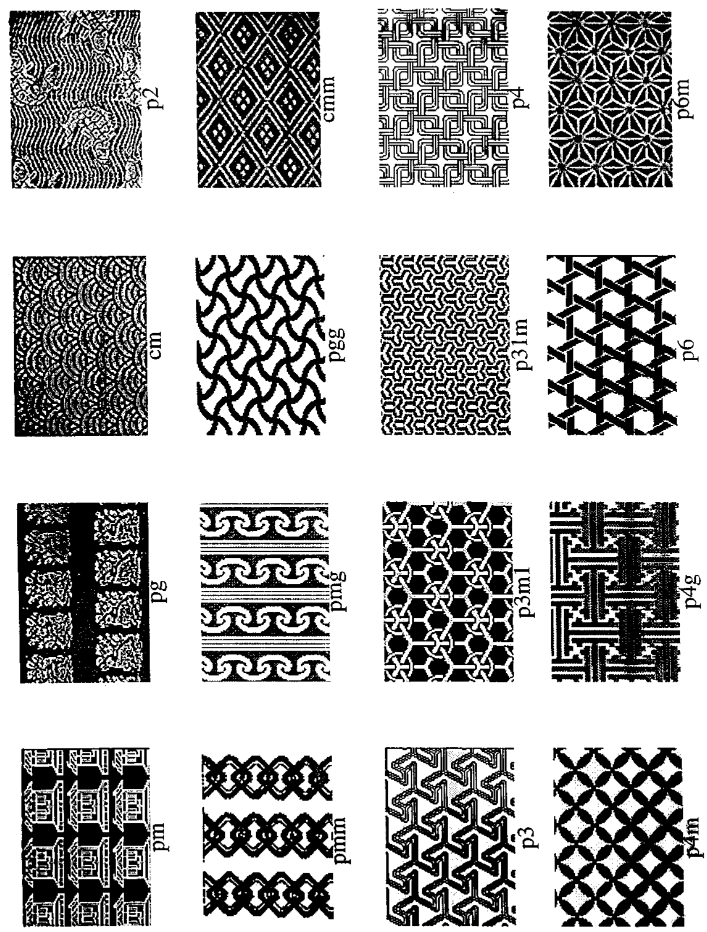
But $|\mathbf{b}| < |\mathbf{a}|$, contradicting our choice of $\mathbf{a}$.

So each rotation in H has order 1, 2, 3, 4 or 6.

(b) Follows from the classification of finite subgroups of $O(2)$. //

Thus each wallpaper group G is built up from a translation subgroup $G \cap T \cong L_G \cong \mathbb{Z} \times \mathbb{Z}$ and a point group $\bar{G} \cong G/(G \cap T) = C_n$ or D_n where $n = 1, 2, 3, 4$ or 6.

From this, it's not hard to see that there are only finitely many possibilities for G. (However, L_G and $\bar{G}$ do not completely determine the group G. Another complication is that a reflection in $\bar{G}$ need not be represented by a reflection in G – could be only represented by glide reflections.)

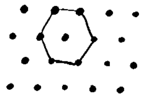


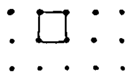
A detailed case-by-case analysis shows that there are exactly 17 wallpaper groups — illustrated in the handout from last time & at the end of the printed notes. (See e.g. Armstrong, chap. 26 for more details.)

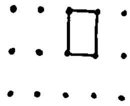
Note:

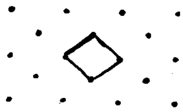
If the point group $\bar{G}$ is "large" this restricts the shape of the lattice L_G.

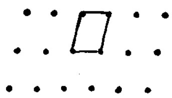
e.g. Point group $\bar{G}$		lattice L_G
C_6, D_6, C_3, D_3	$\Rightarrow$	hexagonal
D_4, C_4	$\Rightarrow$	square
D_2, D_1	$\Rightarrow$	rectangular or rhombic
C_2, C_1 always possible!		take any parallelogram











HIGHER DIMENSIONS:

For a discrete subgroup G of $\mathrm{isom}(\mathbb{E}^n)$, we again have a translational subgroup $G \cap T \subseteq \mathbb{R}^n$, and a point group $\bar{G} \subseteq O(n)$.

For n-dimensional "crystallographic groups", $G \cap T \cong \mathbb{Z}^n = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{n \text{ copies}}$, and $\bar{G}$ is a finite group.

THEOREM (Bieberbach): For any given n, there are only a finite number of crystallographic groups.

In fact there are:

17 groups for $n=2$

230 for $n=3$

4783 for $n=4$!

The 3-dimensional groups are particularly important — they correspond to symmetries of crystals (atoms arranged in a 3-dimensional lattice).