620-231: Vector Analysis

Semester 1, 2008

Problem Sheet 1

Functions of Several Variables

1. Limits for f(x,y). Evaluate the following limits, if they exist:

(a) $\lim_{(x,y)\to(1,0)} x^3 + y^4 + 3$;

(b) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2+2}$;

(c) $\lim_{(x,y)\to(0,0)} \dfrac{3y^2}{2x^2-y^2}$;

(d) $\lim_{(x,y)\to(2,1)} \dfrac{2x^4+3y^3}{3x^2-5}$;

(e) $\lim_{(x,y)\to(0,0)} \dfrac{\cos x - 1 + \frac{x^2}{2}}{x^4+y^4}$;

(f) $\lim_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$;

(g) $\lim_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$;

(h) $\lim_{(x,y)\to(0,0)} \dfrac{y-2x+\sin 2x}{x^3+y}$;

(i) $\lim_{(x,y)\to(1,-1)} \dfrac{x+y}{2x^2+3xy+y^2}$;

(j) $\lim_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$.

2. No Limits. Show that the following limits do not exist:

(a) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{(x^2+y)^3}$;

(b) $\lim_{(x,y)\to(0,0)} \dfrac{2x^4y^4}{(3x^4+y^2)^3}$.

Hint: Examine the limit along $y = x^2$ as well as the usual lines.

3. Limit Exists. Using the Sandwich Theorem, show that

(a) $\lim_{(x,y)\to(0,0)} \dfrac{7x^2y^2}{x^2+2y^4} = 0$;

(b) $\lim_{(x,y)\to(0,0)} \dfrac{3yx^2}{x^2+y^2} = 0$.

4. Continuity. Consider the following functions:

(i) $f(x,y) = ye^x + \sin x$;

(ii) $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$;

(iii) $f(x,y) = \dfrac{xy}{x^2+y^2}$.

(a) Which functions are continuous at $(x,y) = (0,0)$?

(b) Can the functions which are not continuous at $(0,0)$, be made continuous at $(0,0)$ by suitably defining them at the origin?

5. Partial Derivatives Revision. For each function, find the partial derivatives listed.

(a) $f(x,y) = e^x \cos y$. Find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

(b) $f(x,y) = \log(x+3y)$. Find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

(c) $f(x,y) = e^{xy}$. Find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

(d) $f(x,y,z) = x^2y^3z^5$. Find $f_x, f_y, f_z, f_{yy}, f_{yz}, f_{xyz}$.

(e) $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z}$. Find $f_x, f_y, f_z, f_{yz}, f_{yy}, f_{xyy}$.

6. Laplace's Equation. Let

$$V = -\frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

where $(x,y,z) \neq (0,0,0)$ and G, m, M are constants. Show that V satisfies Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

7. Differentiable Functions. Find the domain of each function. Show that the function is C^1 (and is therefore differentiable) in this domain.

(a) $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$; (b) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$; (c) $f(r,\theta) = \dfrac{1}{2} r \sin 2\theta, r > 0$.

8. Continuity and Differentiability. Consider the following function:

$$f(x,y) = \begin{cases} \dfrac{4x^2 - y^3}{x^2 + 5y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) \neq (0,0)$.

(b) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) = (0,0)$.

(c) Is $\dfrac{\partial f}{\partial y}$ continuous at $(0,0)$? Explain.

(d) Is the function C^1 at $(0,0)$?