THE UNIVERSITY OF MELBOURNE

SEMESTER 1, 2006

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-231 VECTOR ANALYSIS

EXAM

Exam duration — 3 hours

Reading time — 15 minutes

This paper consists of 5 pages.

Examination Papers with Common Content:

- This paper contains some questions in common with those for the subject 620-233 Vector Analysis (Advanced) which is being held at the same time.

Instructions to Invigilators:

- Initially, students are to receive a 14 page script book.

Authorized Materials:

- No calculators or computers are permitted.
- No written or printed material may be brought into the examination room.

Instructions to Students:

- There are 10 questions on this examination paper.
- The number of marks allocated to a question are shown below the question.
- The total number of marks for the whole exam is 100.
- All questions may be attempted.
- There is a table of vector identities on page 5 that you may use in this examination.

1. (a) Let $f(x,y) = (2x^2, 3y^2)$, $g(u,v) = u^2 - v^2$ and $h(x,y) = g \circ f$. Calculate $\dfrac{\partial h}{\partial x}$ and $\dfrac{\partial h}{\partial y}$ at $(x,y) = (1,1)$.

   (b) Find the second order Taylor polynomial of

$$u(x,y) = \exp(2x - 3y + 1)$$

   near the point $(1,1)$.

   (c) Let

$$g(x,y) = \begin{cases} \dfrac{2x^3 + y}{x^2 + xy} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   Calculate $\dfrac{\partial g}{\partial x}$ at $(0,0)$.

10 marks

2. Using the method of Lagrange multipliers, find the minimum of the distance squared function $d(x,y,z) = x^2 + y^2 + z^2$ for the points in common with the cone $z^2 = 4(x^2 + y^2)$ and the plane $2y + 4z = 5$.

10 marks

3. (a) Calculate the flow line curve $\mathbf{c}(t)$ to the vector field $\mathbf{F} = (x, -y)$ which passes through the point $(1,2)$.

   (b) Calculate the curvature of the curve

$$\mathbf{c}(t) = \big(\sin t - t\cos t, 2, \cos t + t \sin t\big), \qquad t > 0.$$

10 marks

4. (a) Let $f(x,y,z)$ and $g(x,y,z)$ be any C^1 scalar functions. Show that

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g$$

   (b) Let $\mathbf{r} = (x,y,z)$ and $r = |\mathbf{r}|$. Using vector identities or otherwise, calculate the following quantities where they are defined

   i) $\nabla^2(3r^2)$ ii) $\nabla \times \left(\dfrac{\mathbf{r}}{r^2}\right)$

10 marks

5. (a) Evaluate the integral

$$\iint_D \exp(x+y)\,dx\,dy$$

where D is the domain bounded by the lines $y=0$, $y=x$ and $y=-x+2$.

(b) Consider the integral

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \exp(x^2+y^2)\,dy\,dx$$

i. Sketch the domain of integration, carefully labelling all the boundary curves.

ii. Let $x = u\cos v$ and $y = u\sin v$ be a change of variables. Calculate the Jacobian for this change of variables and hence evaluate the integral.

10 marks

6. Let R be the solid region bounded by the plane $z=0$ and the cone $z = 3 - 3\sqrt{x^2+y^2}$. The z coordinate of the centre of mass is given by

$$z_c = \frac{\iiint_R z\,\mu(x,y,z)\,dV}{\iiint_R \mu(x,y,z)\,dV}$$

where $\mu(x,y,z)$ is the mass density. If $\mu(x,y,z)$ is constant, calculate z_c.

10 marks

7. Let S be a surface parameterised by

$$x = u, \qquad y = v, \qquad z = u^2+v^2, \qquad 0 \le u \le 1, \quad 0 \le v \le 1$$

(a) Describe the curves of constant u.

(b) Find tangent vectors to the lines of constant u and constant v and hence find a normal vector to the surface S.

(c) Find the mass of a thin metal sheet in the shape of the surface S if the mass density per unit area, $m(x,y) = \sqrt{4x^2+4y^2+1}$.

10 marks

8. (a) Let $f(x, y) = 2x^3 + y^2$. Evaluate the path integral

$$\int_{\mathbf{c}} f\, ds$$

where $\mathbf{c}$ is the line $y = -2x + 2$ from $(0, 2)$ to $(1, 0)$.

(b) Let $\mathbf{F} = (2x, -2y, 1)$ be a vector field.

i. Show that $\mathbf{F}$ is a conservative vector field.

ii. Find a function V such that $\nabla V = \mathbf{F}$.

iii. Evaluate the line integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$$

along any curve $\mathbf{c}$, from the point $(0, 0, 0)$ to the point $(1, 1, 1)$.

10 marks

9. Green's theorem can be written as

$$\int_C P\, \mathrm{d}x + Q\, \mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y$$

(a) Verify Green's theorem for the vector field $\mathbf{F} = (x^2, x^2 - y^2)$ and the domain D in common with the two regions $y \leq 1$ and $y \geq x^2$.

(b) If the vector field is changed to $\mathbf{F} = \left(\frac{1}{x}, \frac{1}{y-1} \right)$ then Green's theorem is no longer valid for the domain D. Why?

10 marks

10. Calculate the flux

$$J = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

of the vector field $\mathbf{F} = (0, z, 0)$ across the open surface S consisting of that part of the sphere $x^2 + y^2 + z^2 = 4$ for $x > 0$, $y > 0$ and $z > 0$. Use the normal pointing away from the origin.

10 marks

BASIC IDENTITIES OF VECTOR ANALYSIS

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar functions, $\mathbf{F}$ and $\mathbf{G}$ be vector fields in R^3 and β be any constant.

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla \left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ provided $g \neq 0$.
5. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
6. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\nabla \times (\nabla f) = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$