THE UNIVERSITY OF MELBOURNE

SEMESTER 1, 2007

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-231 VECTOR ANALYSIS

EXAM

Exam duration — 3 hours

Reading time — 15 minutes

This paper consists of 6 pages.

Examination Papers with Common Content:

- This paper contains some questions in common with those for the subject 620-233 Vector Analysis (Advanced) which is being held at the same time.

Instructions to Invigilators:

- Initially, students are to receive a 14 page script book.

Authorized Materials:

- No calculators, mobile phones or computers are permitted.
- No written or printed material may be brought into the examination room.

Instructions to Students:

- There are 10 questions on this examination paper.
- The number of marks allocated to a question are shown below the question.
- The total number of marks for the whole exam is 125.
- All questions may be attempted.
- There is a table of vector identities on page 6 that you may use in this examination.

1. (a) Find the second order Taylor polynomial of

$$g(x, y) = \log(2x + y)$$

about the point $(1, 1)$.

(b) Consider the limit,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2 + 3y^2}.$$

If the limit exists then evaluate it, if it does not exist justify why.

(c) Consider the function,

$$f(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that the partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0, 0)$.

(ii) Is $f(x, y)$ continuous at $(0, 0)$? Justify your answer.

15 marks

2. Using the method of Lagrange multipliers, find the minimum and maximum values of the function

$$h(x, y) = 3x^2 + xy + 6y^2 + 1$$

subject to the constraint $x^2 + 2y^2 = 1$. Justify that the points you have found give the minimum and maximum of h.

10 marks

3. (a) Calculate an expression for the flow line to the vector field $\mathbf{F} = (-x, y)$ which passes through the point $(1, 1)$.

(b) Consider the vector field, $\mathbf{G} = (2y, 2x, z)$.

(i) Is the vector field incompressible? Justify your answer.

(ii) Is the vector field irrotational? Justify your answer.

(iii) Find a function, $\phi(x, y, z)$ such that $\mathbf{G} = \nabla\phi$.

15 marks

4. (a) Let $f(x, y, z)$ be any C^2 scalar function. *Without* using the table of vector identities, show that

$$\nabla \times (\nabla f) = \mathbf{0}.$$

(b) Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Using the table of vector identities or otherwise, calculate the following quantities where they are defined

$$\text{(i)} \quad \nabla \cdot \left(\nabla \times \frac{\mathbf{r}}{r^2} \right) \qquad \text{(ii)} \quad \nabla \cdot (\log(r)\mathbf{r})$$

10 marks

5. (a) Evaluate the integral

$$\iint_D \exp(3x + y)\, dx\, dy$$

where D is the domain bounded by the lines $y = 2x$, $y = x$, $x = 1$ and $x = 3$.

(b) Consider the integral

$$\iint_R x^3 y^2\, dy\, dx$$

where R is the domain bounded by the curves $y = 8x^2$, $y = x^2$, $xy = 1$ and $xy = 2$.

(i) Sketch the domain of integration R, carefully labelling all the boundary curves.

(ii) Let $x = v$ and $y = u/v$ be a change of variables. Use this change of variables to calculate the four boundary curves (in terms of u and v) and hence sketch the new domain, carefully labelling all the boundary curves.

(iii) Calculate the Jacobian for the change of variables in (ii). (Note, do *not* evaluate the integral.)

15 marks

6. Let R be the solid region bounded by the plane $z = 0$ and the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$. Evaluate the triple integral,

$$\iiint_R z\, dx\, dy\, dz.$$

10 marks

7. Let S be a surface parameterised by

$$x = u\cos(v), \qquad y = u, \qquad z = u\sin(v), \qquad 0 \le u \le 1, \quad 0 \le v \le \pi$$

(a) Describe the curves of constant u.

(b) Find tangent vectors to the lines of constant u and constant v and hence find a normal vector to the surface S.

(c) Find the surface area of S.

10 marks

8. (a) Let $f(x, y) = xy$. Evaluate the path integral

$$\int_{\mathbf{c}} f\, ds$$

along the line $y = 2 - 2x$ from the point $(0, 2)$ to the point $(1, 0)$.

(b) Let $\mathbf{F} = (1, -2y, 2z)$ be a vector field.

(i) Is $\mathbf{F}$ a conservative vector field? Justify your answer.

(ii) Evaluate the line integral

$$\int_{\mathbf{c_1}} \mathbf{F} \cdot \mathbf{ds}$$

along the curve $\mathbf{c_1} = (t, t, t^2)$, from the point $A = (0, 0, 0)$ to the point $B = (1, 1, 1)$.

(iii) Using your answer from (ii) deduce the value of the integral

$$\int_{\mathbf{c_2}} \mathbf{F} \cdot \mathbf{ds}$$

along the curve $\mathbf{c_2}(t) = (t, t, t\exp(t^3 - 1))$ from the point $A = (0, 0, 0)$ to the point $B = (1, 1, 1)$. Justify your answer.

10 marks

9. Green's theorem can be written as

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

Verify Green's theorem for the vector field $\mathbf{F} = (-y, x^2)$ and the domain D in common with the regions $0 \leq x \leq 1$ and $0 \leq y \leq x^2$. Include a sketch of D and C and the orientation of C.

15 marks

10. Stokes' theorem can be written as

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \int_{\partial S} \mathbf{F} \cdot \mathbf{ds}.$$

Verify Stokes' theorem for the cone $z = \sqrt{x^2 + y^2}$ with $z \leq 2$ and the vector field

$$\mathbf{F} = (y, 2z, 3x)$$

Include a sketch of S and ∂S, clearly showing the correct normal vector for S and the orientation of ∂S.

15 marks

BASIC IDENTITIES OF VECTOR ANALYSIS

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar functions, $\mathbf{F}$ and $\mathbf{G}$ be vector fields in R^3 and β be any constant.

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ provided $g \neq 0$.
5. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
6. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\nabla \times (\nabla f) = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$