THE UNIVERSITY OF MELBOURNE

SEMESTER 2, 2005

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-231 VECTOR ANALYSIS

Exam duration — 3 hours

Reading time — 15 minutes

This paper consists of 5 pages.

Instructions to Invigilators:

Initially, students are to receive a 14 page script book.

Authorized Materials:

No calculators, computers or mobile phones are permitted.

No written or printed material may be brought into the examination room.

Instructions to Students:

There are 10 questions on this examination paper.

All questions may be attempted.

Marks for each question are indicated on the paper.

The total number of marks on the exam paper is 145.

There are tables of vector identities and general curvilinear coordinates on page 4 and 5, that you may use in this examination.

1. (a) Let $f(x,y,z) = (x+y+z, x^3 - e^{yz})$ and $g(s,t,u) = (st, tu, su)$.

Evaluate the derivative $\mathbf{D}(f \circ g)$ of the composite function $f \circ g$ at $(2,0,-1)$ using the matrix version of the chain rule.

(b) Using the method of Lagrange multipliers, find the maximum and minimum of the function

$$f(x,y,z) = x + y + z$$

subject to the constraints $x^2 + z^2 = 5$ and $2x + y + 3z = 4$. Justify that the points you have found give the maximum and minimum of f.

[20 marks]

2. (a) Calculate the curvature of the path

$$\mathbf{c}(t) = (2\cos 3t, 3t, 2\sin 3t).$$

(b) If the path $\mathbf{u}(t)$ is differentiable at least three times, simplify

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{u}'' \times \mathbf{u}' \cdot \mathbf{u}\right).$$

[12 marks]

3. (a) Let $\mathbf{F}(x,y,z)$ be a C^2 vector field. Prove that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

(b) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Calculate the following quantities, where they are defined:

$$\text{(i)}\nabla \cdot \left(r^3\mathbf{r}\right); \qquad \text{(ii) } \nabla^2\left(\frac{1}{r}\right).$$

[18 marks]

4. Let D be the region enclosed by the lines $y = 0$, $x = 0$ and $y = 1 - x$.

(a) Sketch the region D.

(b) Evaluate

$$\iint_D \cos\left(\frac{x-2y}{x+y}\right)\, dxdy$$

by making the change of variables $u = x - 2y$ and $v = x + y$.

[14 marks]

5. Let R be the solid region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 2 - x^2 - y^2$.

(a) Sketch the region R, including any points of intersection.

(b) Find the total mass of R if the mass density is $(x^2 + y^2)^3$.

[14 marks]

6. Consider the surface parametrized by

$$\Phi(u, v) = (u, u^2 + v, v^2) \qquad -2 \le u \le 2, \quad -2 \le v \le 2.$$

(a) Find a normal vector to the surface in terms of u and v.

(b) Write down a double integral for the area of the surface. Do NOT evaluate this integral.

(c) Find the equation of the tangent plane to the surface at the point $(1, 0, 1)$.

[12 marks]

7. Evaluate

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$$

where

$$\mathbf{F} = \left(2\cos y + e^y + 3x^2, 3\sin x - e^{2x} + y^3\right)$$

and C is the boundary of the rectangle $[-1, 1] \times [0, 3]$ traversed in the clockwise direction. Take $\hat{\mathbf{n}}$ as the unit outward normal to the curve C in the x-y plane.

[8 marks]

8. Let S be the surface of the cone $z = 3 - \sqrt{x^2 + y^2}$ for $z \ge 1$. Evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where $\mathbf{F} = (y + z)\,\mathbf{i} + 3y\mathbf{j} + \left(2x^2 - 4z^3\right)\mathbf{k}$ and S is oriented using the outward unit normal.

[10 marks]

9. Gauss' divergence theorem can be written as

$$\iiint_R \nabla \cdot \mathbf{F}\, dV = \iint_{\partial R} \mathbf{F} \cdot d\mathbf{S}.$$

Verify Gauss' divergence theorem for the vector field $\mathbf{F} = \left(2, 0, z^2\right)$ and the region bounded by the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the plane $z = 0$.

[25 marks]

10. Define orthogonal curvilinear coordinates (u, v, ϕ) by

$$x = uv\cos\phi, \qquad y = uv\sin\phi, \qquad z = \frac{1}{2}(u^2 - v^2)$$

where $u \geq 0, v \geq 0$ and $0 \leq \phi < 2\pi$.

(a) Find the scale factors h_u, h_v, h_ϕ.

(b) Find the unit vectors $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_\phi$.

(c) Find an expression for the volume element.

(d) Evaluate $\nabla^2(u^2\cos\phi)$.

[12 marks]

BASIC IDENTITIES OF VECTOR ANALYSIS

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar functions, $\mathbf{F}$ and $\mathbf{G}$ be vector fields in R^3 and β be any constant.

1. $\nabla(f + g) = \nabla f + \nabla g$

2. $\nabla(\beta f) = \beta\nabla f$

3. $\nabla(fg) = f\nabla g + g\nabla f$

4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ provided $g \neq 0$.

5. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

6. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$

8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$

11. $\nabla \times (\nabla f) = \mathbf{0}$

12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$

13. $\nabla \cdot (\nabla f \times \nabla g) = 0$

14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

IDENTITIES FOR ORTHOGONAL CURVILINEAR COORDINATES

Let $f(u_1, u_2, u_3)$ be a C^2 scalar function and

$$\mathbf{F} = F_1(u_1, u_2, u_3)\mathbf{e_1} + F_2(u_1, u_2, u_3)\mathbf{e_2} + F_3(u_1, u_2, u_3)\mathbf{e_3}$$

be a C^1 vector field. Then

1. $\nabla f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\mathbf{e_1} + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\mathbf{e_2} + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\mathbf{e_3}$

2. $\nabla \cdot \mathbf{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial\left(h_2 h_3 F_1\right)}{\partial u_1} + \dfrac{\partial\left(h_1 h_3 F_2\right)}{\partial u_2} + \dfrac{\partial\left(h_1 h_2 F_3\right)}{\partial u_3}\right]$

3. $\nabla \times \mathbf{F} = \dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e_1} & h_2\mathbf{e_2} & h_3\mathbf{e_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$

4. $\nabla^2 f = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial f}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial f}{\partial u_3}\right)\right]$

END OF EXAMINATION