THE UNIVERSITY OF MELBOURNE

SEMESTER 1, 2008

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-231 VECTOR ANALYSIS

EXAM

Exam duration — 3 hours

Reading time — 15 minutes

This paper consists of 6 pages.

Examination Papers with Common Content:

- This paper contains some questions in common with those for the subject 620-233 Vector Analysis (Advanced) which is being held at the same time.

Instructions to Invigilators:

- Initially, students are to receive a 14 page script book.

Authorized Materials:

- No calculators, mobile phones, MP3 players or computers are permitted.
- No written or printed material may be brought into the examination room.

Instructions to Students:

- There are 10 questions on this examination paper.
- The number of marks allocated to a question are shown below the question.
- The total number of marks for the whole exam is 130.
- All questions may be attempted.
- There is a table of vector identities and coordinate systems on page 6 that you may use in this examination.

1. (a) Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + x^2y^2}{2x^2 + 3y^4}.$$

(i) Evaluate the limit along the path $y = kx$. What does this tell you about the limit?

(ii) Evaluate the limit using the Sandwich theorem.

(b) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2 + y^2}{x + y} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find the partial derivative $\dfrac{\partial f}{\partial x}$ at $(0,0)$.

10 marks

2. Using the method of Lagrange multipliers, find the minimum and maximum values of the function

$$h(x,y,z) = x - xy + 6y + z$$

subject to the constraints $x^2 + 2y = 3$ and $x + 6y + z = 0$. State and justify which points are a minimum and which are a maximum.

12 marks

3. (a) Consider the curve

$$\mathbf{c}(t) = (2\sin t, -2\cos t, -t).$$

Find the unit tangent vector $\mathbf{T}$, the unit principal normal vector $\mathbf{N}$ and the unit binormal vector $\mathbf{B}$ to the curve and hence find the curvature and the torsion of the curve.

(b) Show that the unit principal normal vector of any curve is always perpendicular to the unit tangent vector.

13 marks

4. (a) Let $f(x, y, z)$ be a C^1 scalar function and $\mathbf{F}$ be a C^1 vector field. *Without* using the table of vector identities, show that

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$$

(b) Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Using the table of vector identities or otherwise, calculate the following quantities where they are defined

$$\text{(i)}\nabla \cdot (r^2\mathbf{r}) \qquad \text{(ii)} \quad \nabla \times (r\,\nabla \log r)$$

15 marks

5. Let D be the region enclosed by the lines $y = -x/2$, $y = x$ and $x = 2$.

(a) Sketch the region D carefully labelling all the curves and intersection points.

(b) Calculate the area of the region D using a double integral.

(c) Evaluate the integral

$$\iint_D \cos\left(\frac{x - 2y}{x}\right)\, dx\, dy$$

by making the change of variables $u = x$ and $v = x - 2y$. Sketch the image of D in the uv-plane showing the equations of all three lines.

15 marks

6. Let R be the solid region in common with the region below the surface $z = 4 - x^2 - y^2$ and above the surface $z = 3\sqrt{x^2 + y^2}$. Evaluate the triple integral

$$\iiint_R z\, dV.$$

10 marks

7. (a) Consider the surface parameterised by

$$x = u^2 - v^2, \qquad y = uv - 1 \qquad z = u - v, \qquad 0 \leq u \leq 4, \quad 0 \leq v \leq 4.$$

(i) Find a normal vector to the surface in terms of u and v.

(ii) For what values of u and v is the surface *not* smooth?

(iii) Find the equation of the tangent plane to the surface at $(0, 3, 0)$.

(b) Consider the coordinate system

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv, \qquad z = w.$$

(i) Find the scale factors h_u, h_v and h_w.

(ii) Find the unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$.

(iii) Is this an orthogonal coordinate system? Justify your answer.

15 marks

8. (a) Evaluate the path integral

$$\int_{c_1} x^5 \, ds$$

along the curve $xy = 1$ from $(1, 1)$ to $(\sqrt{2}, 1/\sqrt{2})$.

(b) Let $\mathbf{F} = (x, 2z, 2y)$ be a vector field.

(i) Is $\mathbf{F}$ an irrotational vector field? Justify your answer.

(ii) Is $\mathbf{F}$ an incompressible vector field? Justify your answer.

(iii) Evaluate the line integral

$$\int_{c_2} \mathbf{F} \cdot \mathbf{ds}$$

along the part of the curve $y = x^2$, $z = x$ from $(1, 1, 1)$ to $(2, 4, 2)$.

10 marks

9. The divergence theorem in the plane can be written as

$$\int_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}s = \iint_D \nabla \cdot \mathbf{F} \, \mathrm{d}x \, \mathrm{d}y$$

The domain D is the triangle defined by the three points $(-2, 0)$, $(0, 2)$ and $(0, 0)$. Verify the divergence theorem in the plane for the vector field $\mathbf{F} = (x - y, y)$ and domain D. Include a sketch of D showing the equations of all lines, an example of $\hat{\mathbf{n}}$ on each line and the orientation of ∂D.

15 marks

10. Stokes' theorem can be written as

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \int_{\partial S} \mathbf{F} \cdot \mathbf{ds}.$$

Verify Stokes' theorem for the lower half sphere $x^2 + y^2 + z^2 = 9$, $z \leq 0$ and the vector field

$$\mathbf{F} = (x + y, y, z).$$

Include a sketch of S and ∂S, clearly showing the correct normal vector for S and the orientation of ∂S.

15 marks

BASIC IDENTITIES OF VECTOR ANALYSIS

Let $f(x,y,z)$ and $g(x,y,z)$ be scalar functions, $\mathbf{F}$ and $\mathbf{G}$ be vector fields in R^3 and β be any constant.

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ provided $g \neq 0$.
5. $\nabla \cdot (\mathbf{F}+\mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
6. $\nabla \times (\mathbf{F}+\mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\nabla \times (\nabla f) = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

CYLINDRICAL AND SPHERICAL COORDINATES

Cylindrical coordinates.

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

Spherical coordinates.

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$