Instructions to Invigilators:

Initially, students are to receive a 14 page script book.

Authorized Materials:

No calculators, computers or mobile phones are permitted.
No written or printed material may be brought into the examination room.

Instructions to Students:

There are 12 questions on this examination paper.
All questions may be attempted.
Marks for each question are indicated on the paper.
The total number of marks on the exam paper is 135.
There are tables of vector identities and general curvilinear coordinates on page 5 and 6, that you may use in this examination.

This paper may be reproduced and lodged in the Ballieu Library.
1. (a) Consider the limit

\[ \lim_{(x,y) \to (0,0)} \frac{7x^2y^2}{2x^2 + 5y^4}. \]

(i) Determine the limit when \((x, y)\) approaches \((0, 0)\) along the path \(y = kx\), where \(k \in \mathbb{R}\).

(ii) Using the Sandwich Theorem, evaluate the limit.

(b) Let \(f(u, v) = (uv, 2v, u^2)\) and \(g(x, y, z) = (3x + y, xy + z)\). Evaluate the derivative \(D(f \circ g)\) of the composite function \(f \circ g\) at \((0, 1, -1)\) using the matrix version of the chain rule.

[10 marks]

2. Using the method of Lagrange Multipliers, determine all the real valued critical points of the function

\[ f(x, y, z) = x + 2z \]

subject to the constraints

\[ xz = 2 \quad \text{and} \quad x^2 + y^2 = 1. \]

[10 marks]

3. Consider the path

\[ c(t) = (2t, \sin t, -\cos t). \]

Calculate the unit tangent vector \(T\), the unit principal normal vector \(N\), and the unit binormal vector \(B\), to the path.

[10 marks]

4. (a) Let \(f(x, y, z)\) and \(g(x, y, z)\) be any \(C^2\) scalar functions. Prove identity 13 on page 5 of this exam, namely prove that

\[ \nabla \cdot (\nabla f \times \nabla g) = 0. \]

(b) Let \(r = xi + yj + zk\) and \(r = |r|\). Calculate the following quantities, if they are defined:

(i) \(\nabla \times (r^5r)\); (ii) \(\nabla \cdot (\log_e r)\); (iii) \(\nabla^2 \left(e^{r^2}\right)\).

[15 marks]
5. Let $D$ be the region enclosed by the lines $y = \frac{x}{2}$, $y = 0$ and $y = 1 - x$.
   
   (a) Sketch the region $D$.
   
   (b) Evaluate the double integral
   \[
   \iint_D \sqrt{\frac{x + y}{x - 2y}} \, dA
   \]
   by making the change of variables $u = x + y$ and $v = x - 2y$.

   [12 marks]

6. Let $R$ be the solid region bounded by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Calculate the total mass of $R$ if the mass density is $z^2$.

   [9 marks]

7. Let $S$ be the surface of the paraboloid
   \[
   z = 4 - x^2 - y^2 \quad \text{for} \quad z \geq 0.
   \]
   (a) Determine the outward normal to the surface $S$.
   
   (b) Determine the equation of the tangent plane to the surface at the point $(1, 0, 3)$.
   
   (c) Using part (a), evaluate the surface integral
   \[
   \iint_S x^2 + y^2 + z \, dS.
   \]

   [15 marks]

8. Let $\mathbf{F} = (3x^2y^2 - 6z^3)\mathbf{i} + (2x^3y + 3z)\mathbf{j} + (3y - 18z^2x)\mathbf{k}$.
   
   (a) Show that $\mathbf{F}$ is an irrotational vector field.
   
   (b) Give a physical interpretation of an irrotational vector field. Use a diagram to illustrate your answer.
   
   (c) Determine a scalar function $\phi$ such that $\mathbf{F} = \nabla \phi$.
   
   (d) Evaluate the line integral
   \[
   \int_C \mathbf{F} \cdot d\mathbf{s}
   \]
   along any smooth curve $C$ starting at $(1, 2, 0)$ and finishing at $(0, -1, 1)$.

   [12 marks]

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9. (a) Using Green’s theorem, prove that the area of a domain $D$ in the $x$-$y$ plane is given by the line integral

$$\frac{1}{2} \int_C x \, dy - y \, dx$$

where $C$ is a simple closed curve that bounds $D$.

(b) Using the result in part (a), calculate the area of one loop of the four-leafed rose parametrized by

$$x = \sin 2t \cos t, \quad y = \sin 2t \sin t \quad \text{for} \quad 0 \leq t \leq \frac{\pi}{2}.$$

[10 marks]

10. (a) State Gauss’ Divergence theorem. Draw a diagram to illustrate your statement and explain all symbols used.

(b) Let $S$ be the surface of the rectangular box $[0, 1] \times [0, 2] \times [0, 3]$. Using Gauss’ Divergence theorem, evaluate the surface integral

$$\int \int_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = (x^3 + 5y, y^2 - z^4, 2z - 4y^3)$ and $S$ has an outward pointing normal.

[10 marks]

11. Let $C$ be the closed curve consisting of the semicircle $y = \sqrt{1 - x^2}$ and the line segment $-1 \leq x \leq 1$ traversed in the anticlockwise direction. Let $\mathbf{n}$ be the unit outward normal to the curve $C$ in the $x$-$y$ plane.

(a) Sketch the curve $C$, indicating the direction of $\mathbf{n}$.

(b) Evaluate the path integral

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds$$

where

$$\mathbf{F} = \left(5 \cosh y + e^y + x^3, 4 \sinh x - e^{2x} + y^3\right).$$

[10 marks]

12. Let $S$ be the capped cylindrical surface given by the union of two surfaces $S_1$ and $S_2$ where $S_1$ is $x^2 + y^2 = 1, 1 \leq z \leq 2$ and $S_2$ is $z = 3 - \sqrt{x^2 + y^2}, 2 \leq z \leq 3$.

(a) Sketch the surface $S$.

(b) If $\mathbf{F} = -yz^3\mathbf{i} + 3x\mathbf{j} + x^5\mathbf{k}$, evaluate the surface integral

$$\int \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

with $S$ oriented using the outward unit normal.

[12 marks]
BASIC IDENTITIES OF VECTOR ANALYSIS

Let \( f(x, y, z) \) and \( g(x, y, z) \) be scalar functions, \( \mathbf{F} \) and \( \mathbf{G} \) be vector fields in \( \mathbb{R}^3 \) and \( \beta \) be any constant.

1. \( \nabla (f + g) = \nabla f + \nabla g \)
2. \( \nabla (\beta f) = \beta \nabla f \)
3. \( \nabla (fg) = f \nabla g + g \nabla f \)
4. \( \nabla \left( \frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2} \) provided \( g \neq 0 \).
5. \( \nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} \)
6. \( \nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G} \)
7. \( \nabla \cdot (f \mathbf{F}) = f \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \)
8. \( \nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \)
9. \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \)
10. \( \nabla \times (f \mathbf{F}) = f \nabla \times \mathbf{F} + \nabla f \times \mathbf{F} \)
11. \( \nabla \times (\nabla f) = 0 \)
12. \( \nabla^2 (fg) = f \nabla^2 g + g \nabla^2 f + 2 \nabla f \cdot \nabla g \)
13. \( \nabla \cdot (\nabla f \times \nabla g) = 0 \)
14. \( \nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f \)
Let $f(u_1, u_2, u_3)$ be a $C^2$ scalar function and

$$\mathbf{F} = F_1(u_1, u_2, u_3)\mathbf{e}_1 + F_2(u_1, u_2, u_3)\mathbf{e}_2 + F_3(u_1, u_2, u_3)\mathbf{e}_3$$

be a $C^1$ vector field. Then

1. $\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3$

2. $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 F_1)}{\partial u_1} + \frac{\partial (h_1 h_3 F_2)}{\partial u_2} + \frac{\partial (h_1 h_2 F_3)}{\partial u_3} \right]$

3. $\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 & h_2 & h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$

4. $\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right]$