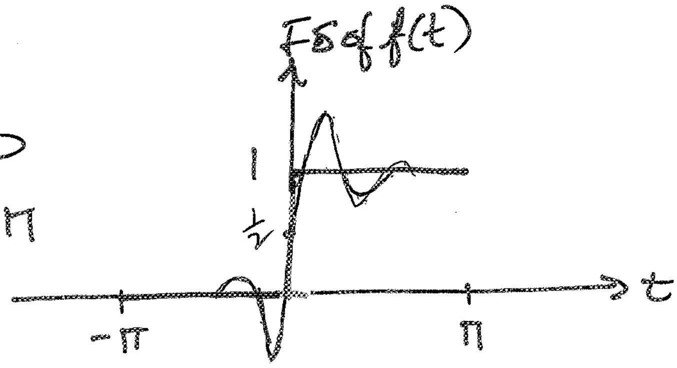
Gibbs Phenomena

Function

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$



$T = \pi$

Fourier Series

$$\frac{a_0}{2} = \frac{1}{2T}\int_{-T}^{T} f(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dt + \int_{0}^{\pi} 1\,dt\right]$$

$$= \frac{\pi}{2\pi} = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos\frac{n\pi t}{\pi}\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin\frac{n\pi t}{\pi}\,dt$$

$$= \frac{-1}{n\pi}\cos nt \Big|_0^{\pi}$$

$$= -\frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \frac{-1}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$$

$\therefore$ let $n = 2m+1$ for $m = 0, 1, 2 \ldots$

$$\therefore S(t) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{2}{(2m+1)\pi} \sin(2m+1)t$$

<u>Finite</u> Fourier Series : N terms

$$S_N(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{N-1} \frac{\sin(2m+1)t}{2m+1}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{N-1} \int_0^t \cos(2m+1)x \, dx$$

$$= \frac{1}{2} + \frac{2}{\pi} \int_0^t \sum_{m=0}^{N-1} \cos(2m+1)x \, dx$$

- OK to interchange the Σ and $\int$ as the series is finite

<u>Sum</u> <u>Recall</u> $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

so $$\sum_{m=0}^{N-1} \cos(2m+1)x = \sum_{m=0}^{N-1} \frac{e^{i(2m+1)x} + e^{-i(2m+1)x}}{2}$$

$$= \frac{1}{2}\left(\sum_{m=0}^{N-1} e^{ix} e^{2mix} + e^{-ix} e^{-2mix}\right)$$

$$= \frac{1}{2} e^{ix} \sum_{m=0}^{N-1} e^{2mix} + \frac{1}{2} e^{-ix} \sum_{m=0}^{N-1} e^{-2mix}$$

Recall $S_N = \sum_{m=0}^{N-1} r^m$ (geometric series)

$$= 1 + r + r^2 + \dots + r^{N-1}$$

$$r S_N = r + r^2 + r^3 + \dots + r^N$$

$$S_N - r S_N = 1 - r^N$$

$$S_N (1-r) = 1 - r^N$$

$$S_N = \frac{1-r^N}{1-r}$$

so $\sum_{m=0}^{N-1} e^{2mix} = \sum_{m=0}^{N-1} \left(e^{2ix}\right)^m$ $\quad [e^{2ix} = r]$

$$= \frac{1 - \left(e^{2ix}\right)^N}{1 - e^{2ix}}$$

$$= \frac{1 - e^{2iNx}}{1 - e^{2ix}}$$

also $\sum_{m=0}^{N-1} e^{-2mix} = \sum_{m=0}^{N-1} \left(e^{-2ix}\right)^m$

$$= \frac{1 - \left(e^{-2ix}\right)^N}{1 - e^{-2ix}}$$

$$= \frac{1 - e^{-2iNx}}{1 - e^{-2ix}}$$

get $\sum_{m=0}^{N-1} \cos(2m+1)x = \frac{1}{2} e^{ix}\left(\frac{1-e^{2iNx}}{1-e^{2ix}}\right) + \frac{1}{2} e^{-ix}\left(\frac{1-e^{-2iNx}}{1-e^{-2ix}}\right)$

$$= \frac{1}{2} e^{iNx} \frac{\sin Nx}{\sin x} + \frac{1}{2} e^{-iNx} \frac{\sin Nx}{\sin x}$$

$$= \frac{1}{2} \frac{\sin Nx}{\sin x}\left(e^{iNx} + e^{-iNx}\right)$$

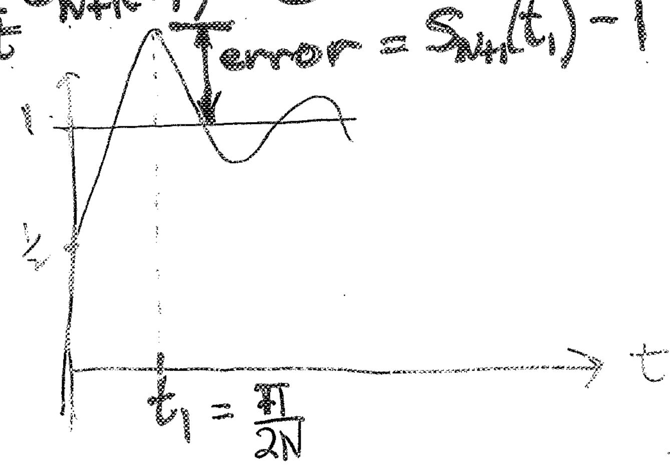
$$= \frac{\sin Nx}{\sin x} \cos Nx$$

$$= \frac{\sin 2Nx}{2 \sin x}$$

- So $S_{N+1}(t) = \frac{1}{2} + \frac{2}{\pi} \int_0^t \frac{\sin 2Nx}{2 \sin x}\, dx$

We want to calculate the error in $S_{N+1}(t)$ at the first peak when $N+1 \longrightarrow \infty$

- Find t_1 where $\frac{d}{dt} S_{N+1}(t_1) = 0$



Now $e^{ix}\dfrac{(1-e^{2iNx})}{1-e^{2ix}} = \dfrac{e^{ix}(1-e^{iNx})(1+e^{iNx})}{(1-e^{ix})(1+e^{ix})}$

$$= \frac{e^{ix}e^{iNx}(e^{-iNx}-1)(1+e^{iNx})}{e^{ix}(e^{-ix}-1)(1+e^{ix})}$$

$$= e^{iNx}\frac{(e^{-iNx}-e^{iNx})}{(e^{-ix}-e^{ix})}$$

$$= e^{iNx}\frac{-(e^{iNx}-e^{-iNx})}{-(e^{ix}-e^{-ix})}$$

$$= e^{iNx}\frac{\left(\frac{e^{iNx}-e^{-iNx}}{2i}\right)}{\left(\frac{e^{ix}-e^{-ix}}{2i}\right)}$$

$$= e^{iNx}\frac{\sin Nx}{\sin x}$$

Also $e^{-xi}\dfrac{(1-e^{-2iNx})}{1-e^{-2ix}} = e^{-iNx}\dfrac{\sin Nx}{\sin x}$

$$\frac{dS_{N+1}(t)}{dt} = \frac{1}{\pi}\,\frac{\sin 2Nt}{\sin t} = 0$$

ie $2Nt = \pi, 2\pi, 3\pi, \ldots$

$$t = \frac{\pi}{2N}, \frac{\pi}{N}, \frac{3\pi}{2N}, \ldots$$

first minimum t

first maximum t_1

- Error as $N \to \infty$

$$\equiv S_{N+1}\left(\frac{\pi}{2N}\right) - 1$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^{\frac{\pi}{2N}} \frac{\sin 2Nx}{\sin x}\,dx - 1$$

let $u = 2Nx$

$$\frac{du}{dx} = 2N$$

and $x = \frac{u}{2N}$

$$= -\frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin u}{\sin\left(\frac{u}{2N}\right)} \cdot \overbrace{\frac{1}{2N}}^{\frac{dx}{du}}\,du$$

Now as $N \to \infty$ $\sin\left(\frac{u}{2N}\right) \to \frac{u}{2N}$ (standard limits)

$\therefore$ get $\text{Error} \equiv S_{N+1}\left(\frac{\pi}{2N}\right) - 1$

$$\approx -\frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin u}{\frac{u}{2N}} \cdot \frac{1}{2N}\, du$$

$$= -\frac{1}{2} + \frac{1}{\pi}\underbrace{\int_0^{\pi} \frac{\sin u}{u}\, du}$$

$$= -\frac{1}{2} + 0.5895$$ via Numerical integration

$$= 0.0895$$

$$\sim 9\%$$