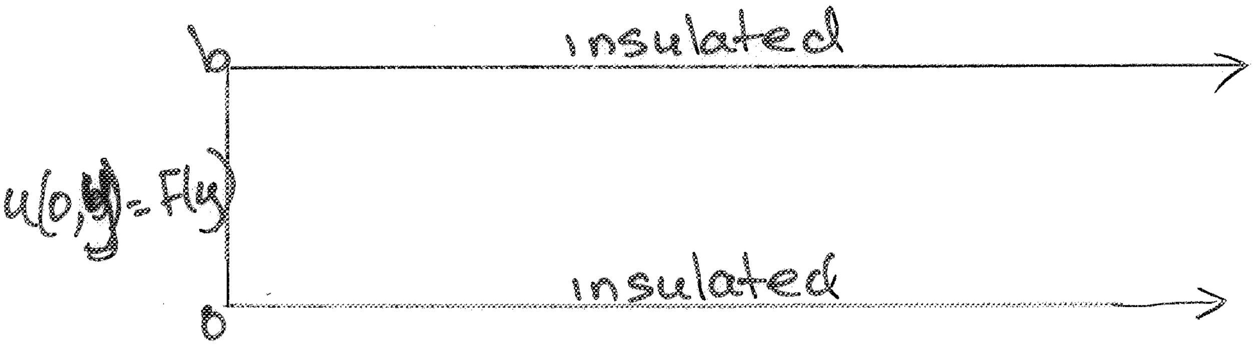
Example 8: A semi-infinite plate - (steady state)



Steady state
Laplace's eqn $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < \infty, \quad 0 < y < b$

$\dfrac{\partial u}{\partial y}(x,0) = 0 \qquad \dfrac{\partial u}{\partial y}(x,b) = 0$

$$u(0,y) = F(y)$$

also as $x \to \infty$ $\quad u$ is finite

Same procedure as per example 7. Still try

$$u(x,y) = X(x)\,Y(y)$$

get $\quad \lambda = 0 \qquad Y_0(y) = m \qquad m \in \mathbb{R}$

$\qquad \lambda > 0 \qquad Y_n(y) = C_n \cos \dfrac{n\pi y}{b}$

However solving for $X(x)$ in

$\lambda > 0$ $$\frac{d^2X}{dx^2} - \lambda X = 0 \qquad \lambda_n = \left(\frac{n\pi}{b}\right)^2, n = 1,2,3 ..$$

$$0 < x < \infty$$

is now of the form

$$X(x) = p_n e^{-\frac{n\pi x}{b}} + q_n e^{\frac{n\pi x}{b}}$$

b/c of the boundedness constraint

on $u(x \to \infty, y)$

As $x \to \infty$ $e^{\frac{n\pi x}{b}} \to \infty$ an unrealistic situation!!

$$\Rightarrow q_n = 0$$

$$\therefore X_n(x) = p_n e^{-\frac{n\pi x}{b}}$$

Also

$\lambda = 0$ get $X_0(x) = Ax + B$

but as $x \to \infty$ $X_0(x) \to \infty$

$$\Rightarrow A = 0$$

$$\therefore X_0(x) = B$$

Now get

$$u(x,y) = \underbrace{Bm}_{\substack{X(x)Y(y) \\ \lambda = 0}} + \sum_{n=1}^{\infty} \underbrace{p_n e^{-\frac{n\pi x}{b}} C_n \cos\frac{n\pi y}{b}}_{\substack{X_n(x)\,Y_n(y) \\ \lambda > 0}}$$

so $u(x,y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi x}{b}} \cos\frac{n\pi y}{b}$

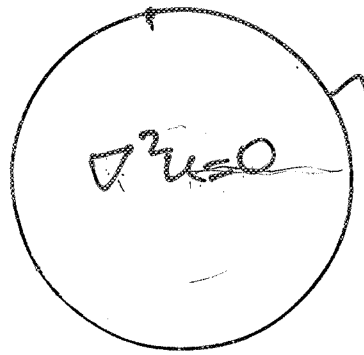
A Fourier Cosine Series again $\boxed{T=b}$

$$u(0,y) = F(y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi y}{b}$$

so $\frac{A_0}{2} = \frac{\cancel{2}}{2b}\int_0^b F(y)\,dy$

$$A_n = \frac{2}{b}\int_0^b F(y)\cos\frac{n\pi y}{b}\,dy$$

Example 9 Circular BBQ (radius 1) ← steady state again



$u(1,\theta) = f(\theta)$

Need to use polar coordinates b/c circular

diffusion/heat equation in 2D

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

becomes $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

[As $t \to \infty$ $\dfrac{\partial u}{\partial t} = 0$ for a realistic situation]

<u>Note</u>: For polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$\therefore$ $\nabla^2 u = 0$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad *$$

$0 < r < 1$

$-\pi < \theta < \pi$