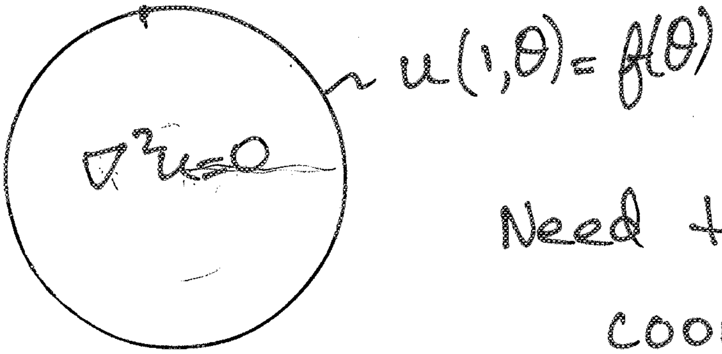
Example 9 Circular BBQ (radius 1) ← steady state again



Need to use polar coordinates b/c circular

diffusion/heat equation in 2D

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

becomes $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

[As $t \to \infty$ $\dfrac{\partial u}{\partial t} = 0$ for a realistic situation]

Note: For polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$\therefore$ $\nabla^2 u = 0$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad *$$

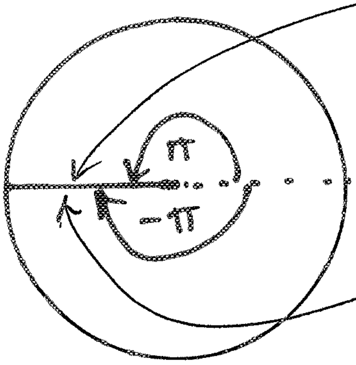
$0 < r < 1$

$-\pi < \theta < \pi$

Boundary Conditions

$u(1,\theta) = f(\theta)$ was given

but we must deduce the other BCs



temperature and flux are continuous

$\therefore u(r,-\pi) = u(r,\pi)$

$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$

Trial solution

$$u(r,\theta) = R(r)\,G(\theta)$$

$\frac{\partial u}{\partial r} = \frac{dR}{dr} G \qquad \frac{\partial^2 u}{\partial \theta^2} = R \frac{d^2 G}{d\theta^2}$

substituting gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r G \frac{dR}{dr}\right) + \frac{1}{r^2} R \frac{d^2 G}{d\theta^2} = 0$$

multiply by $\frac{r^2}{GR}$

$$\frac{r^2}{GRr}\frac{\partial}{\partial r}\left(r G \frac{dR}{dr}\right) + \frac{R}{r^2}\cdot\frac{r^2}{GR}\frac{d^2 G}{d\theta^2} = 0$$

$$\frac{r}{R}\frac{d}{dr}\left(r \frac{dR}{dr}\right) + \frac{1}{G}\frac{d^2 G}{d\theta^2} = 0$$

so $-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = +\frac{1}{G}\frac{d^2G}{d\theta^2} = -\lambda$

$$\Rightarrow \quad \frac{d^2G}{d\theta^2} + \lambda G = 0 \qquad -\pi < \theta < \pi$$

$u(r,-\pi) = R(r)G(-\pi) \qquad u(r,\pi) = R(r)G(\pi)$

and $u(r,-\pi) = u(r,\pi)$

$$\Rightarrow \quad R(r)G(-\pi) = R(r)G(\pi)$$

$$\Rightarrow \quad G(-\pi) = G(\pi)$$ b/c $R(r) = 0$ gives a trivial solution

also $\frac{\partial u(r,-\pi)}{\partial\theta} = R(r)G'(-\pi)$

$\frac{\partial u}{\partial\theta}(r,\pi) = R(r)G'(\pi)$

and $\frac{\partial u}{\partial\theta}(r,-\pi) = \frac{\partial u}{\partial\theta}(r,\pi)$

$$\Rightarrow \quad R(r)G'(-\pi) = R(r)G'(\pi)$$

$$\Rightarrow \quad G'(-\pi) = G'(\pi)$$ b/c $R(r) = 0$ gives a trivial solution

BVP :

$$\boxed{\begin{aligned} &\frac{d^2G}{d\theta^2} + \lambda G = 0 \qquad -\pi < \theta < \pi \\ &G(-\pi) = G(\pi) \quad , \quad G'(-\pi) = G'(\pi) \end{aligned}}$$

Test the 3 cases of λ yourselves

$\underline{\lambda > 0}$

eigen values $\lambda_n = n^2 \quad n = 1, 2, 3 \ldots$

eigen functions $G_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$

eigenvalue $\lambda = 0$

eigenfunction $G_0(\theta) = C$ const

The R equation

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \lambda \qquad --- (2)$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{\lambda R}{r} = 0$$

$\lambda = 0$ or n^2

$$\underbrace{r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \frac{\lambda R}{r} = 0}_{\text{Equidimensional Equation}} \qquad --- (3)$$

$0 < r < a$

No Homogeneous Boundary Conditions!

For $\underline{\lambda = 0}$ it is easier to solve

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \lambda \qquad --(2)$$

get $\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$

so $r\frac{dR}{dr} = k \qquad k \in \mathbb{R}$

$$\frac{dR}{dr} = \frac{k}{r}$$

$$R = k\operatorname{Log}|r| + l \qquad l \in \mathbb{R}$$

As $r \to 0$ $R(r)$ needs to be finite

$\Rightarrow k = 0$

For $\underline{\lambda > 0}$ $(\lambda = n^2)$ it is easier to

solve $r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \frac{\lambda R}{r} = 0 \quad --(3)$

Here we try $R(r) = r^p$

$$\therefore R'(r) = pr^{p-1}$$

$$R''(r) = p(p-1)r^{p-2}$$

+ substitute into (3)

$$rp(p-1)r^{p-2} + pr^{p-1} - \frac{n^2r^p}{r} = 0$$

$$p(p-1)r^{p-1} + pr^{p-1} - n^2r^{p-1} = 0$$

Now $r^{p-1} \neq 0$ so

$$p(p-1) + p - n^2 = 0$$

$$p^2 - n^2 = 0$$

$$p = \pm n$$

$$\therefore \quad R(r) = R_n(r) = c_n r^n + d_n r^{-n} \quad c_n, d_n \in \mathbb{R}$$

But as $r \to 0$ we need $R(r)$ to be finite (realistic)

$$\Rightarrow \quad d_n = 0$$

$$\therefore R(r) = R_n(r) = C_n r^n$$

Put solutions together

$\underline{\lambda = 0}$

$$\underbrace{u(r,\theta)}_{\lambda=0} = \underset{\lambda=0}{R(r)}\ \underset{\lambda=0}{G(\theta)}$$

$$= 1\,C = \frac{a_0}{2}$$

$\underline{\lambda > 0}$

$$u_n(r,\theta) = \underset{\lambda>0}{R_n(r)}\ \underset{\lambda>0}{G_n(\theta)}$$

$$= C_n r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$\therefore u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

(linear superposition)