Example If $f(t) = t\cos t$ find $F(s)$

$$F(s) = \int_0^\infty e^{-st}\, t \cos t \, dt$$

Integration by parts Yuk!!

Try another way

demonstrated in lecture

In general LT of $t f(t)$

$$\mathcal{L}\{t f(t); t \to s\} = \int_0^\infty e^{-st} t \, f(t)\,dt$$

$$= \int_0^\infty -\frac{d}{ds} e^{-st} f(t)\,dt$$

$$= -\frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

$$= -\frac{d}{ds} F(s)$$

Find $\mathcal{L}\{t^2 f(t) ; t \to s\}$.

$$\mathcal{L}\{t^2 f(t) ; t \to s\} = \int_0^\infty e^{-st} t^2 f(t)\,dt$$

$$= \int_0^\infty -\frac{d}{ds} e^{-st} \, t f(t)\,dt$$

$$= -\frac{d}{ds} \int_0^\infty e^{-st} t \, f(t)\,dt$$

$$= -\frac{d}{ds} \int_0^\infty -\frac{d}{ds} e^{-st} f(t)\,dt$$

$$= \frac{d^2}{ds^2} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{d^2}{ds^2} F(s)$$

Example

Find the ILT of $\dfrac{s}{(s^2+4)^2}$

(done in lecture)

2. LT of derivatives

can be used to solve ODEs, PDEs

$$\mathcal{L}\{f'(t)\,;t\to s\} = \int_0^\infty e^{-st} f'(t)\,dt \quad \text{integrate by parts}$$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty (-s)\,e^{-st} f(t)\,dt$$

Require $f(t)\,e^{-st} \to 0$ as $t\to\infty$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \underbrace{-f(0)}_{\text{initial value of } f} + sF(s)$$

Similarly for the n^{th} derivative $f^{n}(t)$

$$\mathcal{L}\{f^{n}(t)\,;t\to s\} = \int_0^\infty e^{-st} f^{n}(t)\,dt$$

$$= f^{(n-1)}(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} f^{(n-1)}(t)\,dt$$

$$= -f^{(n-1)}(0) + s\,\mathcal{L}\{f^{(n-1)}(t)\,;t\to s\}$$

This is a recurrence relation

$n=1 \quad \mathcal{L}\{f'(t)\,;t\to s\} = sF(s) - f(0)$

$n=2 \quad \mathcal{L}\{f''(t)\,;t\to s\} = s\mathcal{L}\{f'(t)\} - f'(0)$

$$= s\big(sF(s) - f(0)\big) - f'(0)$$

$$= s^2F(s) \underbrace{- sf(0) - f'(0)}_{\text{2 initial conditions}}$$

$n=3 \quad \mathcal{L}\{f'''(t)\,;t\to s\} = s\mathcal{L}\{f''(t)\} - f''(0)$

$$= s\big(s^2F(s) - sf(0) - f'(0)\big) - f''(0)$$

$$= s^3F(s) \underbrace{- s^2f(0) - sf'(0) - f''(0)}_{\text{3 initial conditions}}$$

(where "may" require

$f(0) = \lim_{t\to 0^+} f(t)$ etc

3. Solving ODEs by LT

Given ODE in $f(t)$ with initial conditions

$$\Big\downarrow \mathcal{L}$$

Algebraic equation in $F(s)$. Solve

$$\Big\downarrow \mathcal{L}^{-1}$$

Get Solution $f(t)$

<u>Example</u>: Using LT solve for $f(t)$, $t > 0$

$$f''(t) - 9f(t) = 9t \quad *$$

where $f(0) = 1$, $f'(0) = 0$

(done in lecture)

You are <u>strongly</u> advised to check your answer by finding $f''(t)$ and substituting everything into $*$

Check the initial conditions too.

Example $\quad f'' - f' - 6f = t \qquad (\dagger)$

where $f(0) = 1$, $f'(1) = 4$ or $f(1) = 6$

solve $(\dagger)$

mixed data

(not an initial value problem)

Let $A = f'(0)$

Working during lecture gave

$$F(s) = \frac{\frac{1}{s^2} + s + (A-1)}{(s-3)(s+2)}$$

$$= \frac{1}{s^2(s-3)(s+2)} + \frac{s+(A-1)}{(s-3)(s+2)}$$

end of lect 31