

Example If $f(t) = t \cos t$ find $F(s)$

$$F(s) = \int_0^{\infty} e^{-st} t \cos t \, dt$$

Integration by parts yuk!!

Try another way

demonstrated in lecture

In general LT of $t f(t)$

$$\mathcal{L}\{t f(t); t \rightarrow s\} = \int_0^{\infty} e^{-st} t f(t) dt$$

$$= \int_0^{\infty} -\frac{d}{ds} e^{-st} f(t) dt$$

$$= -\frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt$$

$$= -\frac{d}{ds} F(s)$$

Find $\mathcal{L}\{t^2 f(t); t \rightarrow s\}$.

$$\mathcal{L}\{t^2 f(t); t \rightarrow s\} = \int_0^{\infty} e^{-st} t^2 f(t) dt$$

$$= \int_0^{\infty} -\frac{d}{ds} e^{-st} t f(t) dt$$

$$= -\frac{d}{ds} \int_0^{\infty} e^{-st} t f(t) dt$$

$$= -\frac{d}{ds} \int_0^{\infty} -\frac{d}{ds} e^{-st} f(t) dt$$

$$= \frac{d^2}{ds^2} \int_0^{\infty} e^{-st} f(t) dt$$

$$= \frac{d^2}{ds^2} F(s)$$

Example

Find the ILT of $\frac{s}{(s^2+4)^2}$

(done in lecture)

2. LT of derivatives

can be used to solve ODEs, PDEs

$$\begin{aligned} \mathcal{L}\{f'(t); t \rightarrow s\} &= \int_0^{\infty} e^{-st} f'(t) dt && \text{integrate} \\ & && \text{by parts} \\ &= f(t) e^{-st} \Big|_0^{\infty} - \int_0^{\infty} (-s) e^{-st} f(t) dt \end{aligned}$$

Require $f(t) e^{-st} \rightarrow 0$ as $t \rightarrow \infty$

$$= -f(0) + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + s F(s)$$

∩

initial value of f

Similarly for the n^{th} derivative $f^{(n)}(t)$

$$\mathcal{L}\{f^{(n)}(t); t \rightarrow s\} = \int_0^{\infty} e^{-st} f^{(n)}(t) dt$$

$$= f^{(n-1)}(t) e^{-st} \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} f^{(n-1)}(t) dt$$

$$= -f^{(n-1)}(0) + s \mathcal{L}\{f^{(n-1)}(t); t \rightarrow s\}$$

This is a recurrence relation

$$n=1 \quad \mathcal{L}\{f'(t); t \rightarrow s\} = sF(s) - f(0)$$

$$n=2 \quad \mathcal{L}\{f''(t); t \rightarrow s\} = s \mathcal{L}\{f'(t)\} - f'(0)$$

$$= s(sF(s) - f(0)) - f'(0)$$

$$= s^2 F(s) - \underbrace{s f(0) - f'(0)}$$

2 initial conditions

$$n=3 \quad \mathcal{L}\{f'''(t); t \rightarrow s\} = s \mathcal{L}\{f''(t)\} - f''(0)$$

$$= s(s^2 F(s) - s f(0) - f'(0)) - f''(0)$$

$$= s^3 F(s) - \underbrace{s^2 f(0) - s f'(0) - f''(0)}$$

3 initial conditions

where "may" require

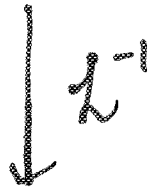
$$f(0) = \lim_{t \rightarrow 0^+} f(t) \quad \text{etc}$$

3. Solving ODEs by LT

Given ODE in $f(t)$ with
initial conditions



Algebraic equation
in $F(s)$. Solve



Get solution $f(t)$

Example: Using LT solve for $f(t)$, $t > 0$

$$f''(t) - 9f(t) = 9t \quad *$$

where $f(0) = 1$, $f'(0) = 0$

(done in lecture)

You are strongly advised to check your answer by finding $f''(t)$

and substituting everything into *

Check the initial conditions too.

Example $f'' - f' - 6f = t \quad (t)$

where $f(0) = 1$, $f'(1) = 4$ or $f(1) = 6$
solve (t) ↑ ↘
mixed data

Let $A = f'(0)$ (not an initial value problem)

Working during lecture gave

$$F(s) = \frac{\frac{1}{s^2} + s + (A-1)}{(s-3)(s+2)}$$
$$= \frac{1}{s^2(s-3)(s+2)} + \frac{s+(A-1)}{(s-3)(s+2)}$$

end of lect 31