Chapter 3: Linear Programming

Problems

These are optimization problems whose objective functions and constraints are linear.

Definition 3.1.1

A real-valued function $f$, defined on $\Omega \subseteq \mathbb{R}^n$ is a linear function if and only if there exists a $c \in \mathbb{R}^n$ such that

$$f(x) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n.$$
Definition 3.1.2

A linear equation is an expression of the form

\[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = b \]

A linear inequality is an expression of the form

\[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \leq b \]

or

\[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \geq b. \]
Example 3.1.3

\[ z^* := \max_x 4x_1 + 3x_2 \]

\[ x_1 + x_2 \leq 30 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
3.2 Formulation of a Linear Program

We are moving towards a convenient form for expressing linear programming problems. Let

- $m$ denote the number of functional constraints
- $n$ denote the number of decision variables
- $b_i$ denote the available level of resource $i$, $i = 1, 2, 3, ..., m$
- $x_j$ denote the level of activity $j$, $j = 1, 2, 3, ..., n$
- $z$ denote the value of objective function
- $c_j$ denote the return/cost per unit of activity $j$
- $a_{ij}$ denote the amount of resource consumed /produced by each unit of activity $j$. 
3.3 Assumptions of LP Problems

- Constant return to scale: the return $c_j x_j$ is a fixed multiple of the activity level $x_j$

- Continuous activity level. For activity $j$, the activity level $x_j$ is a continuous variable. However, quite often we approximate a problem where the activity levels must be integers by a formulation where the activities are continuous variables. Strictly speaking, in such cases it is necessary to justify the use of the approximation.

- Independence of activities

\[
f(x_1, \ldots, x_n) = c_1 x_1 + \ldots + c_j x_j + \ldots + c_n x_n
\]
\[
a_{i,1} x_1 + \ldots + a_{i,j} x_j + \ldots + a_{i,n} x_n \leq b_i
\]
For activity $j$, we have

- Decision $x_j$
- Constraints
  
  $a_1 j x_j$
  $a_2 j x_j$
  ...
  ...
  $a_m j x_j$

- Return/cost $c_j x_j$. 
And we end up with

\[ \text{opt}_{x z} = \sum_{j=1}^{n} c_j x_j \]

\[ \sum_{j=1}^{n} a_{ij} x_j \sim b_i, i = 1, \ldots, m \]

\[ x_j \geq 0, j = 1, \ldots, n \]

where

\[ \sim \in \{\leq, =, \geq\} \]
Standard Form

- \( opt = \text{max} \)
- \( \sim \leq \)
- \( b_i \geq 0, \text{ for all } i \)
Thus we have

\[
\max_{x} z = \sum_{j=1}^{n} c_j x_j
\]

\[
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1
\]

\[
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2
\]

\[
\ldots
\]

\[
a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m
\]

\[
x_j \geq 0, j = 1, \ldots, n
\]
3.4 Example

Apex Corporation produces canned beans. It has two canneries:

- one can produce up to 400 cans per day
- one can produce up to 600 cans per day
It also has three warehouses

- one needs at least 300 cans per day
- one needs at least 200 cans per day
- one needs at least 400 cans per day
The distances between canneries and warehouses are given (in some unit of distance).

<table>
<thead>
<tr>
<th></th>
<th>W’house 1</th>
<th>W’house 2</th>
<th>W’house 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cannery 1</td>
<td>75</td>
<td>40</td>
<td>85</td>
</tr>
<tr>
<td>Cannery 2</td>
<td>20</td>
<td>55</td>
<td>75</td>
</tr>
</tbody>
</table>
Task

Formulate a linear programming model to determine a minimum cost transportation schedule that satisfies the warehouses needs and does not exceed the production capacities of the canneries.
Analysis

- Decision Variables:
  Let $x_{ij} =$ number of cans shipped from cannery $i$ to warehouse $j$, $i = 1, 2$; $j = 1, 2, 3$

- Objective function:

$$f(x) := 75 \times x_{11} + 40 \times x_{12} + 85 \times x_{13}$$
$$+ 20 \times x_{21} + 55 \times x_{22} + 75 \times x_{23}$$
• **Constraints:**
  
  **Production capacities:**

\[
\begin{align*}
  x_{11} + x_{12} + x_{13} &\leq 400 \text{(cannery 1)} \\
  x_{21} + x_{22} + x_{23} &\leq 600 \text{(cannery 2)}
\end{align*}
\]
• Warehouse requirements:
\[x_{11} + x_{21} \geq 300 (\text{warehouse 1})\]
\[x_{12} + x_{22} \geq 200 (\text{warehouse 2})\]
\[x_{13} + x_{23} \geq 400 (\text{warehouse 3})\]

• Non-negativity:
\[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0.\]

or, perhaps, even:
\[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \in \{0, 1, 2, 3, \ldots\}\]
Complete Formulation

\[ z^* := \min z \]
\[ = 75x_{11} + 40x_{12} + 85x_{13} + 20x_{21} + 55x_{22} + 75x_{23} \]

such that

\[ x_{11} + x_{12} + x_{13} \leq 400 \]
\[ x_{21} + x_{22} + x_{23} \leq 600 \]
\[ x_{11} + x_{21} \geq 300 \]
\[ x_{12} + x_{22} \geq 200 \]
\[ x_{13} + x_{23} \geq 400 \]
\[ x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0 \]