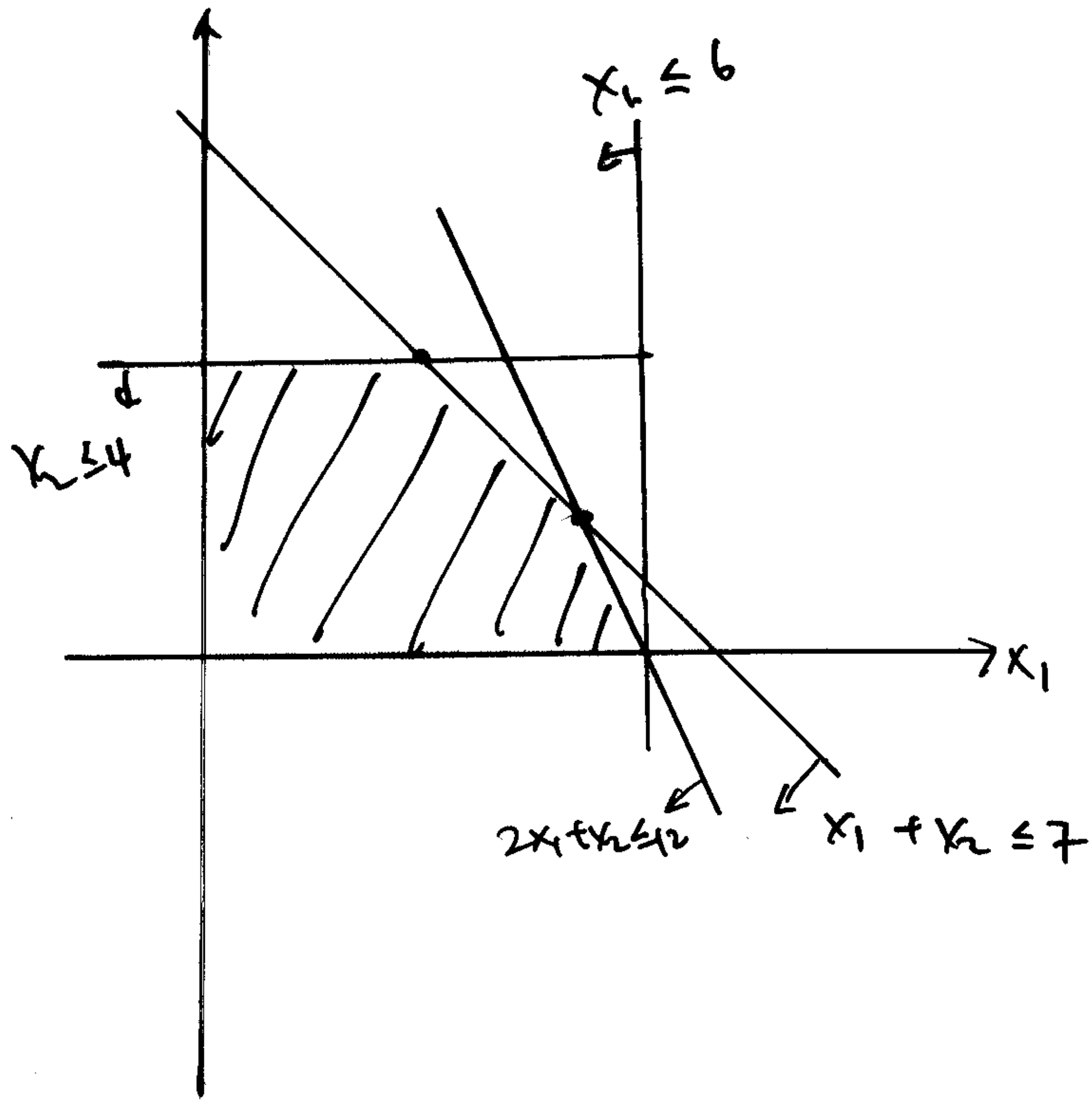
24/10 ①



Problem:

$$\max \{ x_1 + 2x_2 ,\ x_1 - x_2 \}$$
$$\text{s.t.} \quad x_1 + x_2 \le 7$$
$$2x_1 + x_2 \le 12$$
$$x_1 \le 6$$
$$x_2 \le 4$$
$$x_1, x_2 \ge 0$$

$z = \{ x_1 + 2x_2, \quad x_1 - x_2 \}$

$z(\beta) = x_1 + 2x_2 + \beta (x_1 - x_2)$

$= (1+\beta)x_1 + (2-\beta)x_2.$

$\boxed{\text{when } \beta = 0}$. $\rightarrow$ $\boxed{z(\beta) = x_1 + 2x_2}$

Optimising w.r.t. $x_1 + 2x_2$.

Initial tableau:

BV	x_1	x_2	S_1	S_2	S_3	S_4	RHS
S_1	1	1	1	0	0	0	7
S_2	2	1	0	1	0	0	12
S_3	1	0	0	0	1	0	6
S_4	0	1	0	0	0	1	4
r_j	-1	-2	0	0	0	0	0

Final tableau:

BV	x_1	x_2	S_1	S_2	S_3	S_4	RHS
x_1	1	0	1	0	0	-1	3
S_2	0	0	-2	1	0	1	2
S_3	0	0	-1	0	1	1	3
x_2	0	1	0	0	0	1	4
r_j	0	0	1	0	0	1	11

$x^* = (3, 4)$, $\quad z^*(\beta)\big|_{\beta=0} = 3 + 2(4)$
$= 11.$

?? For what values of β does the same ext. pt. $x^* = (3,4)$ gives the optimal z^*?

$x_1 + 2x_1 + \underline{\beta}(x_1 - x_2)$

$\boxed{\text{Range Analysis}}$

BV	x_1	x_2	s_1	s_2	s_3	s_4	RHS	
x_1	1	0	1	0	0	-1	3	
s_2	0	0	-2	1	0	1	2	← leave
s_3	0	0	-1	0	1	1	3	
x_2	0	1	0	0	0	1	4	
r_j	$-\beta$	β	1	0	0	1	11	
(row x_1)$\times\beta$ + (row r_j)	0	β	$\beta+1$	0	0	$-\beta+1$	$3\beta+11$	
(row x_2)$\times(-\beta)$ + row(r_j)	0	0	$\beta+1$	0	0	$-2\beta+1$	$-\beta+11$	
						↑ enter		

λz $\qquad \lambda > 0 \qquad \max\{z_1, z_2\}$

$\max \lambda z$

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Optimal for $\beta+1 \geq 0$ and $-2\beta+1 \geq 0$

from $\beta+1 \geq 0 \Rightarrow \beta \geq -1$ (but we assume $\beta \geq 0$ anyway)

from $-2\beta+1 \geq 0$ we get $\beta \leq \frac{1}{2}$.

So, $z^*(\beta) = 11 - \beta$ for $0 \leq \beta \leq \frac{1}{2}$ at ext. point $x^* = (3, 4)$.

x^* no longer optimal when $\beta > \frac{1}{2}$. Hence critical value of $\beta = \beta^* = \frac{1}{2}$.

If $\beta > \frac{1}{2}$, reduced cost for S_4 becomes -ve, so S_4 enters the basis, and by ratio test S_2 leaves.

Restoring canonical form, we get:

BV	x_1	x_2	S_1	S_2	S_3	S_4	RHS
x_1	1	0	-1	1	0	0	5
S_4	0	0	-2	1	0	1	2
S_3	0	0	1	-1	1	0	1 ←
x_2	0	1	2	-1	0	0	2
r_j	0	0	$-3\beta+3$	$2\beta-1$	0	0	$3\beta+9$

Range analysis:

Sol$^{\underline{n}}$ is optimal for $-3\beta+3 \geq 0 \Rightarrow \beta \leq 1$

and $+2\beta-1 \geq 0 \Rightarrow \beta \geq \frac{1}{2}$.

Ie. Sol$^{\underline{n}}$ opt. for $\frac{1}{2} \leq \beta \leq 1$.

$x^* = (5,2)$ and $z^*(\beta) = 9+3\beta$.

Critical value $\beta^* = 1$.

For $\beta>1$, s_1 has a negative reduced cost, so S_1 enters the basis and by ratio test, S_3 leaves

BV	x_1	x_2	S_1	S_2	S_3	S_4	RHS
x_1	1	0	0	0	1	0	6
S_4	0	0	0	-1	2	1	4
S_1	0	0	1	-1	1	0	1
x_2	0	1	0	1	-2	0	0 ←
r_j	0	0	0	$-\beta+2$	$3\beta-3$	0	$6\beta+6$

Sol$^{\underline{n}}$ opt for $-\beta+2 \geq 0 \Rightarrow \beta \leq 2$

$3\beta-3 \geq 0 \Rightarrow \beta \geq 1$

Critical pt. $\beta^* \geq 2$.

When $\beta > 2$, S_2 enters the basis, and x_2 leaves.

BV	x_1	x_2	S_1	S_2	S_3	S_4	RHS
x_1	1	0	0	0	1	0	6
S_4	0	0	0	-1	0	1	4
S_1	0	1	1	0	-1	0	1
S_2	0	1	0	1	-2	0	0
r_j	0	$\beta-2$	0	0	$\beta+1$	0	$6\beta+6$

$x^* = (6, 0)$ $\quad z^*(\beta) = 6\beta + 6$.

Soln opt. for $\beta \geq 2$.

Critical pt. $\beta^* = +\infty$. STOP.

$\therefore$ Extreme pts. $x^* = \{(3,4), (5,2), (6,0)\}$.

Pareto Efficient Extreme pts.

$z^* = \{(11,-1), (9,3), (6,6)\}$.